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A TURBULENT BOUNDARY-LAYER CALCULATION METHOD BASED ON THE LAW OF THE WALL AND THE LAW OF THE WAKE

by

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ABSTRACT. A method for calculating the incompressible turbulent boundary layer based on the "law of the wall" and Coles' "law of the wake" is presented. The method is applicable to two-dimensional bodies and to bodies of revolution in axisymmetric flow when the boundary-layer thickness is not necessarily small compared to the body radius. The carrying out of this method involves a simultaneous solution of the momentum integral equation and the energy integral equation, assuming that the mean velocity profiles are given by a universal two-parameter representation as suggested by Coles.

Since these two-parameter, mean velocity profiles inherently contain sufficient information to determine the shear stress at the wall and the rate of energy dissipation in the boundary layer, and accurately represent velocity profiles to the point of separation, it is anticipated that there will result an improved technique for estimating the growth of a turbulent boundary layer which may be valid to the point of separation.

U. S. NAVAL ORDNANCE TEST STATION

China Lake, California

November 1964

U. S. NAVAL ORDNANCE TEST STATION

AN ACTIVITY OF THE BUREAU OF NAVAL WEAPONS

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FOREWORD

This report presents the theoretical development of a method for calculating the incompressible turbulent boundary layer based on the "law of the wall" and the "law of the wake." This development was carried out to provide a more rigorous solution of the boundary-layer equations for turbulent flow, in order to obtain an improved technique for estimating the inflow velocity profiles to propellers and the skin-friction drag of bodies of revolution.

The work was done at the U.S. Naval Ordnance Test Station between November 1962 and January 1964 under Bureau of Naval Weapons Task Assignment RUTO-3E-000/216-1/F008-06-02 Problem Assignment 231. H. E. Eggers was the Bureau of Naval Weapons Project Engineer.

The considered opinions of the Propulsion Division are presented in this report.

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CONTENTS

NOMENCLATURE

A	Profile parameter (Eq. 13)
C	An empirical constant in the "law of the wall" (Eq. 4)
e	Unit vector
G	Shape parameter (Eq. 7)
h	Scale factor
H_r	Thickness ratio, δ_r^*/θ_r
$\overline{H}_r$	Thickness ratio, δ_r^{**}/θ_r
K	Curvature of body surface in plane of mean flow
ℓ	Mixing length
L	A length defined by Eq. 27
m	Numerical constant equal to 2.75
n	Numerical constant equal to 2.50
N_{1-38}	Numerical constants (see Appendix)
O	Order of magnitude
p	Static pressure of the fluid
$\overline{p}$	Mean static pressure
p'	Fluctuation static pressure
r	Radial coordinate (Fig. 3)
r_o	Body radius
R_e	Reynolds number
u, v, w	Components of fluid velocity in direction of increasing x, y, γ, respectively
$\overline{u}, \overline{v}, \overline{w}$	Mean values of fluid velocity components
u', v', w'	Fluctuation values of fluid velocity components
U	Fluid velocity at edge of boundary
U_o	Undisturbed free-stream velocity
v_*	$\sqrt{\tau_o/\rho}$
$\vec{V}$	Velocity of fluid
$W(y/\delta)$	"Law of the wake" function

x, y, γ	Orthogonal curvilinear coordinates (Fig. 3)
X	A length used to define a Reynolds number
β	Angle between tangent to body surface and axis of symmetry
δ	Boundary-layer thickness
δ_c	See discussion, page 11
δ^*	Displacement thickness
δ_r^*	Pseudo-displacement thickness for $1/r_o \neq 0$; displacement thickness for $1/r_o = 0$
δ_r^{**}	Pseudo-energy thickness for $1/r_o \neq 0$; energy thickness for $1/r_o = 0$
Δ	Turbulent thickness (Eq. 6)
θ	Momentum thickness
θ_r	Pseudo-momentum thickness for $1/r_o \neq 0$; momentum thickness for $1/r_o = 0$
κ	Proportionality between mixing length and distance from the wall
ν	Kinematic viscosity
Π	Profile parameter (Eq. 5)
ρ	Fluid density
τ	Shear stress
τ_o	Shear stress at the wall
τ_ℓ	Laminar shear stress
τ_t	Turbulent shear stress

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INTRODUCTION

Coles' boundary-layer investigations (Ref. 1) showed that the two-dimensional, incompressible, turbulent boundary-layer, mean velocity profiles in the whole range from transition to separation closely approximate a two-parameter family. This fact suggests the possibility of using these profiles in place of the more common power-law profiles[1] in computing the development of the turbulent boundary layer. Such a method might be expected to yield satisfactory results up to the point of separation, whereas power-law profile methods tend to break down near separation partly because at this point the mean velocity profiles cannot be accurately approximated by a power law.

Since one of the parameters governing the two-parameter family is $v_* = \sqrt{\tau_o/\rho}$, the shear stress at the wall is specified once the mean velocity profile is specified. Hence, a method that uses these two-parameter velocity profiles would not have to make use of an approximate expression, such as a flat-plate formula, for the shear at the wall. These profiles also allow the rate at which energy is dissipated in the boundary layer to be calculated. In general, their use permits a more rigorous approach to the boundary-layer solution. Furthermore, an investigation based on available experimental data (detailed below) indicates that the mean velocity profiles on bodies of revolution in axisymmetric flow also fall reasonably well within this two-parameter family so that such a method may be applied almost equally well to a body in axisymmetric flow.

Using Coles' two-parameter velocity profile family as a starting point, the method presented here for determining the incompressible turbulent boundary layer on two-dimensional bodies and bodies of revolution in axisymmetric flow was undertaken. Since the equations governing the axisymmetric flow about a body of revolution reduce to those for the flow over a two-dimensional body as $r_o \to \infty$ (r_o = body radius), only the axisymmetric development is given. By setting $1/r_o = 0$ at any point in the axisymmetric development, the equivalent two-dimensional result is obtained.

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[1] Power-law profiles are of the form $\overline{u}/U = (y/\delta)^k$.

HISTORICAL REVIEW

In 1925, Prandtl (Ref. 2) made an important contribution to the theory of turbulent flow in the development of his mixing-length hypothesis. For the case of near-parallel two-dimensional flow, this hypothesis results in a relationship between the shearing stress, the mixing length, and the gradient of the velocity normal to the direction of the flow.

$$\tau = \rho \ell^2 \left(\frac{\partial \bar{u}}{\partial y} \right)^2 \tag{1}$$

Applying this result to the flow near a wall, Prandtl assumed that the mixing length varied proportionally with the distance from the wall and that the shearing stress was independent of the distance from the wall.

$$\begin{aligned} \ell &= \kappa y \\ \tau &= \tau_o = \rho v_*^2 \end{aligned} \tag{2}$$

The first of these assumptions follows from the fact that the mixing length must go to zero at the wall. The second assumption is reasonable if the pressure gradient in the direction of flow is small, since for near-parallel two-dimensional flow,

$$\frac{1}{\rho} \frac{\partial \bar{p}}{\partial x} \cong \frac{\partial \tau}{\partial y}$$

Substituting Eq. 2 into Eq. 1 and integrating from $\bar{u} = 0$ to $\bar{u} = \bar{u}$, there is obtained

$$\frac{\bar{u}}{v_*} = \frac{1}{\kappa} (\ln y - \ln y_o) \tag{3}$$

where y_o is the unknown distance from the wall,[2] where $\bar{u} = 0$. Using a dimensional argument, y_o must be proportional to ν / v_*. Hence Eq. 3 takes the following form:

$$\frac{\bar{u}}{v_*} = \frac{1}{\kappa} \ln \frac{y v_*}{\nu} + C \tag{4}$$

This is the logarithmic mean velocity distribution which is the "law of the wall" for two-dimensional, incompressible flow.

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[2] Since Eq. 3 is not applicable to the region very near the wall where the laminar stresses predominate (the laminar sublayer), it does not satisfy the boundary condition $\bar{u} = 0$ where $y = 0$.

The validity of Eq. 4 for flow past a wall was substantiated by the experiments in 1940 of Schultz-Grunow (Ref. 3). In these experiments, the mean velocity profiles and skin friction were measured for turbulent flow over a flat plate with no pressure gradient. The data from the inner part of the boundary layer was found to be in good agreement with Eq. 4.

An additional result of these experiments was that the deviation of the velocity profiles from Eq. 4 in the outer part of the boundary layer could be expressed quite accurately as a function of y/δ only. Thus the mean velocity profiles for turbulent flow over a flat plate with no pressure gradient could be expressed as

$$\frac{\bar{u}}{v_*} = \frac{1}{\kappa} \ln \frac{yv_*}{\nu} + C + F\left(\frac{y}{\delta}\right)$$

The fact that the Schultz-Grunow experiments were carried out with no pressure gradient made them consistent with the assumption used in developing Eq. 4 that the $\partial\tau/\partial y$ is very small. The question therefore arose whether Eq. 4 would also be valid for boundary layers developing in a strong pressure gradient where the $\partial\tau/\partial y$ would no longer be small. This question was answered by the experiments in 1949 of Ludwieg and Tillman (Ref. 4), in which the mean velocity profiles and skin friction were measured for turbulent flow over a flat plate in both positive and negative pressure gradients. Once again the data from the inner part of the boundary layer (excluding the laminar sublayer and transition zone) were found to be in good agreement with Eq. 4. However, the deviation of the velocity profiles from Eq. 4 in the outer part of the boundary layer was found to be no longer only a function of y/δ but also a function of the distance along the plate, x. Hence, the mean velocity profiles for turbulent flow over a flat plate with a pressure gradient could be expressed as

$$\frac{\bar{u}}{v_*} = \frac{1}{\kappa} \ln \frac{yv_*}{\nu} + C + F\left(\frac{y}{\delta}, x\right)$$

In 1956, Coles (Ref. 1) investigated the form of the function describing the deviation from Eq. 4 of the velocity profiles in the outer portion of the boundary layer. He examined essentially all the experimental data pertinent to the growth of two-dimensional turbulent boundary layers in a pressure gradient, and determined that, to a very close approximation, this function of y/δ and x could be expressed as the product of a function only of x and a function only of y/δ. Thus Coles expressed the mean velocity profiles for two-dimensional turbulent boundary layers in the following form:

$$\frac{\bar{u}}{v_*} = \frac{1}{\kappa} \ln \frac{yv_*}{\nu} + C + \frac{\Pi(x)}{\kappa} W\left(\frac{y}{\delta}\right) \tag{5}$$

Coles refers to the function $W(y/\delta)$ as the "law of the wake" since it represents the portion of the flow in the boundary layer that is characteristic of wake flow, i.e., essentially unaffected by τ_0. The function $W(y/\delta)$ is given in Table 1. The profile parameter, Π, together with v_* determines the shape of the mean velocity profiles.

TABLE 1. Coles' Wake Function

y/δ	W	y/δ	W
0.00	0.000	0.55	1.152
0.05	0.004	0.60	1.307
0.10	0.029	0.65	1.458
0.15	0.084	0.70	1.600
0.20	0.168	0.75	1.729
0.25	0.272	0.80	1.840
0.30	0.396	0.85	1.926
0.35	0.535	0.90	1.980
0.40	0.685	0.95	1.999
0.45	0.838	1.00	2.000
0.50	0.994		

THE TURBULENT THICKNESS AND SHAPE PARAMETER

The fitting of analytical expressions for velocity profiles to experimental data is often done by requiring equal momentum thicknesses, θ, and equal displacement thicknesses, δ^*. However, for an expression of the form of Eq. 5 it is much more convenient not to work with θ and δ^* directly but to make use of the turbulent thickness, Δ, and the shape parameter, G, utilized by Clauser (Ref. 5).

$$\Delta = \int_0^\delta \frac{U - \bar{u}}{v_*} dy \qquad (6)$$

$$G = \int_0^{\delta/\Delta} \left(\frac{U - \bar{u}}{v_*}\right)^2 d\left(\frac{y}{\Delta}\right) \qquad (7)$$

It can easily be shown that two velocity profiles having equal values of Δ, G, and v_* also have equal values of θ and δ^*.

DETERMINATION OF v_* FROM MEAN VELOCITY PROFILE

Because of the apparent universal applicability of Eq. 4 to the inner part (excluding the laminar sublayer and transition zone) of the turbulent boundary layer, it may be used to make a reasonable estimate of the value of v_*/U if only the mean velocity profile is known. Eq. 4 may be written as follows:

$$\frac{\overline{u}}{U} = \frac{v_*}{U}\frac{1}{\kappa}\ln y + \frac{v_*}{U}\frac{1}{\kappa}\ln\frac{v_*}{\nu} + \frac{v_*}{U}C$$

Taking the derivative with respect to ln y,

$$\frac{v_*}{U} = \kappa\frac{d(\overline{u}/U)}{d(\ln y)} \tag{8}$$

By plotting $\overline{u}/U$ versus ln y and determining the slope of the linear (or near-linear) inner portion of the profile, the value of v_*/U is obtained.

APPLICABILITY OF COLES' RESULT TO THE BOUNDARY LAYER ON A BODY OF REVOLUTION IN AXISYMMETRIC FLOW

The boundary-layer equations for the axisymmetric flow over a body of revolution reduce to the equations for flow over a two-dimensional body as $\delta/r_o \rightarrow 0$. Hence, the velocity profiles over the forward part of a body of revolution in axisymmetric flow where $\delta/r_o << 1$ should be almost identical to those found on a two-dimensional body. However, over the after part of the body where δ/r_o is no longer small, the deviation of the velocity profiles in the outer portion of the boundary-layer from the law of the wall, Eq. 4, might be expected to depend on δ/r_o in such a fashion that Coles' result, Eq. 5, for two-dimensional flow might not apply. To test whether Eq. 5 could adequately represent the mean velocity profiles for the axisymmetric case, the measured mean velocity profiles (Ref. 6, 7, 8) on the after part of several bodies of revolution were compared with Eq. 5.

The comparison was carried out in the following manner. Each profile was plotted as $\overline{u}/U$ versus ln y and an estimate of v_*/U was determined from Eq. 8. This value of v_*/U and the curve[3] of $\overline{u}/U$ versus

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[3] Due to the lack of reliable data very near the wall, the measured profiles were extended analytically to the wall using Eq. 4 and the equation for the flow in the laminar sublayer, $u/v_* = yv_*/\nu$.

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y were used in Eq. 6 and 7 to determine values of Δ and G by numerical integration. Applying the boundary condition $\bar{u} = U$ when $y = \delta$ to Eq. 5, there is obtained

$$\frac{U}{v_*} = \frac{1}{\kappa} \ln \frac{\delta v_*}{\nu} + C + \frac{\Pi}{\kappa} W(1) \tag{9}$$

Subtracting Eq. 5 from Eq. 9 yields,

$$\frac{U - \bar{u}}{v_*} = -\frac{1}{\kappa} \ln \frac{y}{\delta} + \frac{\Pi}{\kappa}\left[W(1) - W\left(\frac{y}{\delta}\right)\right] \tag{10}$$

Substituting Eq. 10 into Eq. 6 and 7 and carrying out the integration,[4] the terms involving $W(y/\delta)$ being integrated numerically using Table 1, there results

$$\delta = \frac{\kappa \Delta}{1 + \Pi}$$

$$1.522\Pi^2 + (3.200 - \kappa G)\Pi + 2 - \kappa G = 0 \tag{11}$$

Equations 11 permit values of δ and Π to be determined from the experimentally obtained values of Δ and G. Putting these values of δ and Π into Eq. 5 rewritten in the following form,

$$\frac{\bar{u}}{U} = \frac{v_*}{U} \frac{1}{\kappa} \ln\left(\frac{y v_* U}{X U U_o} R_e\right) + \frac{v_*}{U} C + \frac{v_*}{U} \frac{\Pi}{\kappa} W\left(\frac{y}{\delta}\right)$$

where

$$R_e = \frac{X U_o}{\nu}$$

a direct comparison on the basis of $\bar{u}/U$ versus y can be made between the experimental profiles and Eq. 5. The comparisons are shown in Fig. 1a - d. The agreement is sufficiently good that application of Coles' result for two-dimensional flow, Eq. 5, to a body of revolution in axisymmetric flow appears reasonably sound.

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[4] Since Eq. 5 or Eq. 10 does not hold in the region very close to the wall (the laminar sublayer and the transition zone), the terms involving $\ln(y/\delta)$ give rise to improper integrals. These improper integrals are, however, convergent and the deviation of Eq. 5 or Eq. 10 from the true flow near the wall can be shown to have an insignificant effect on the values of parameters such as Δ and G.

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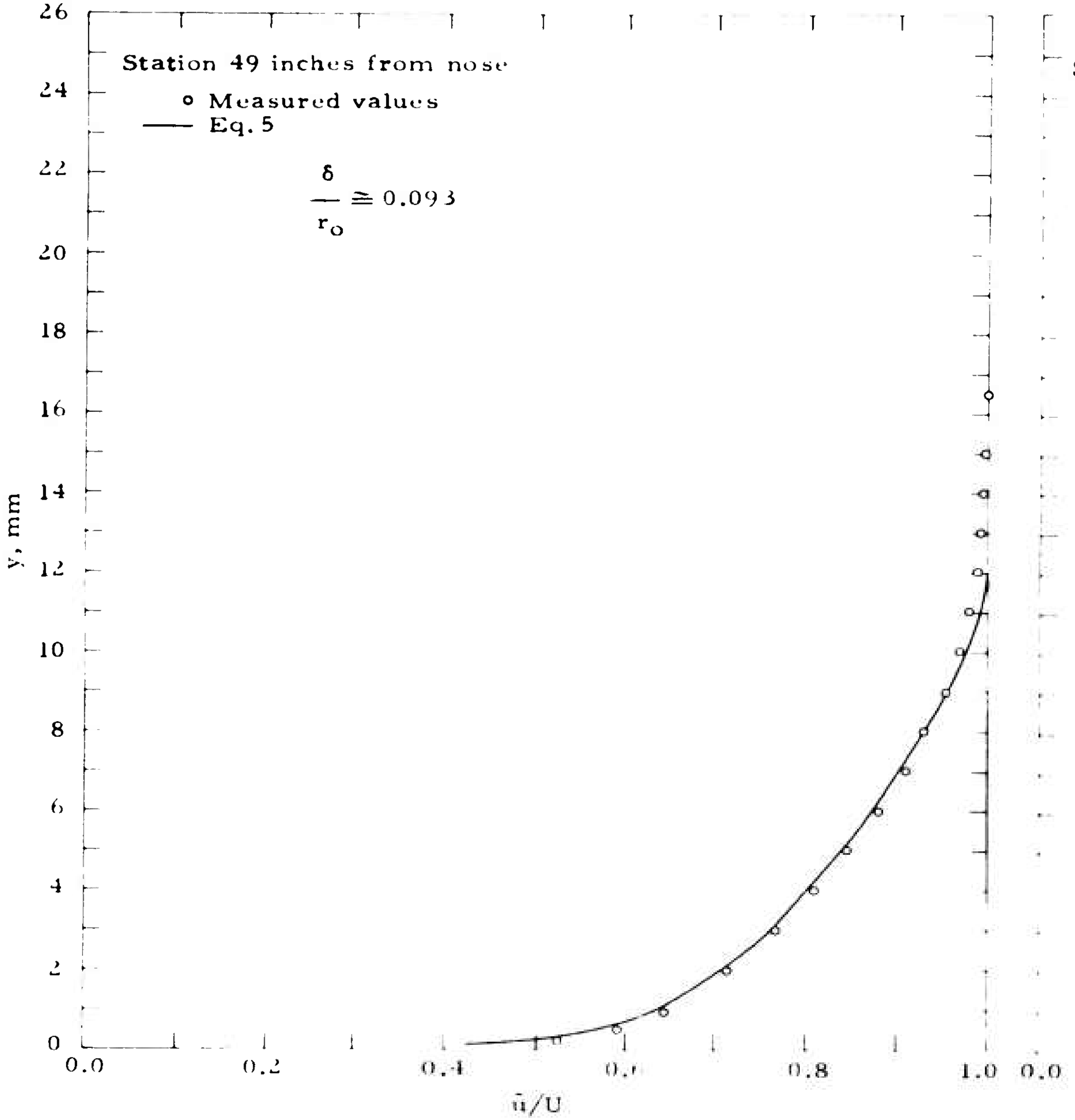
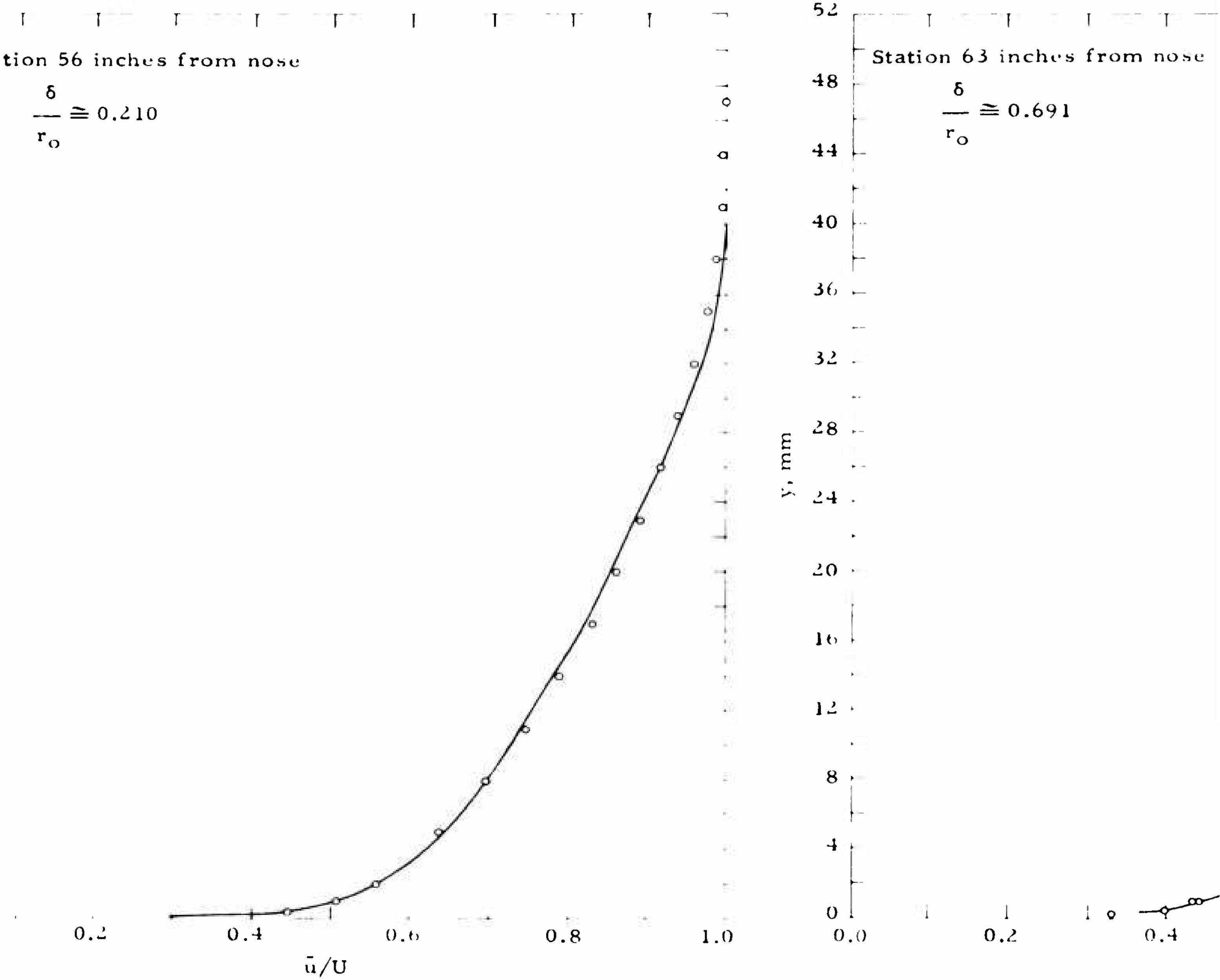


FIG. 1. Equation 5



ompared With Boundary-Layer Measurements Made by Several Investigators.
(a) Lyon (Ref. 6), Body A.

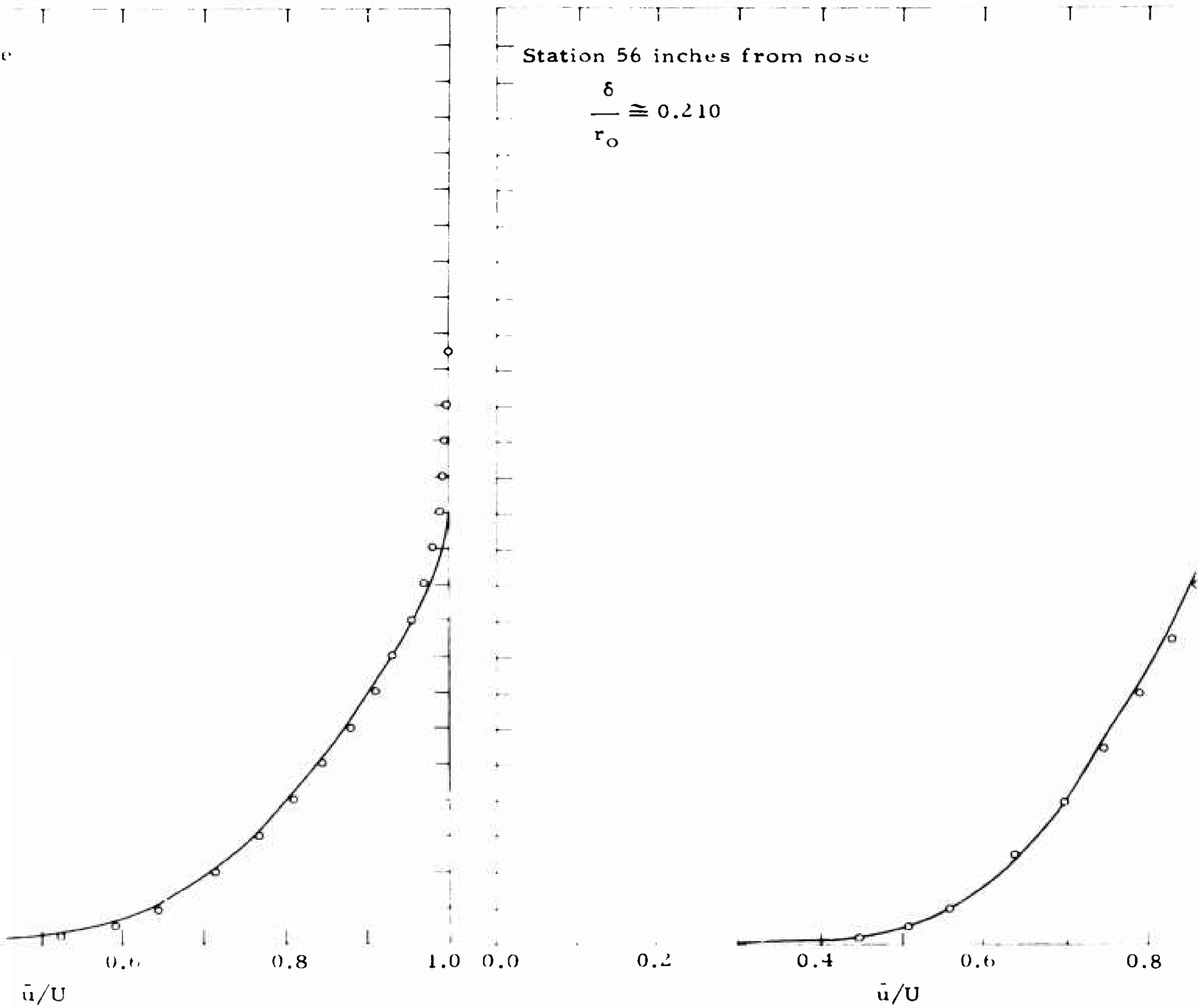
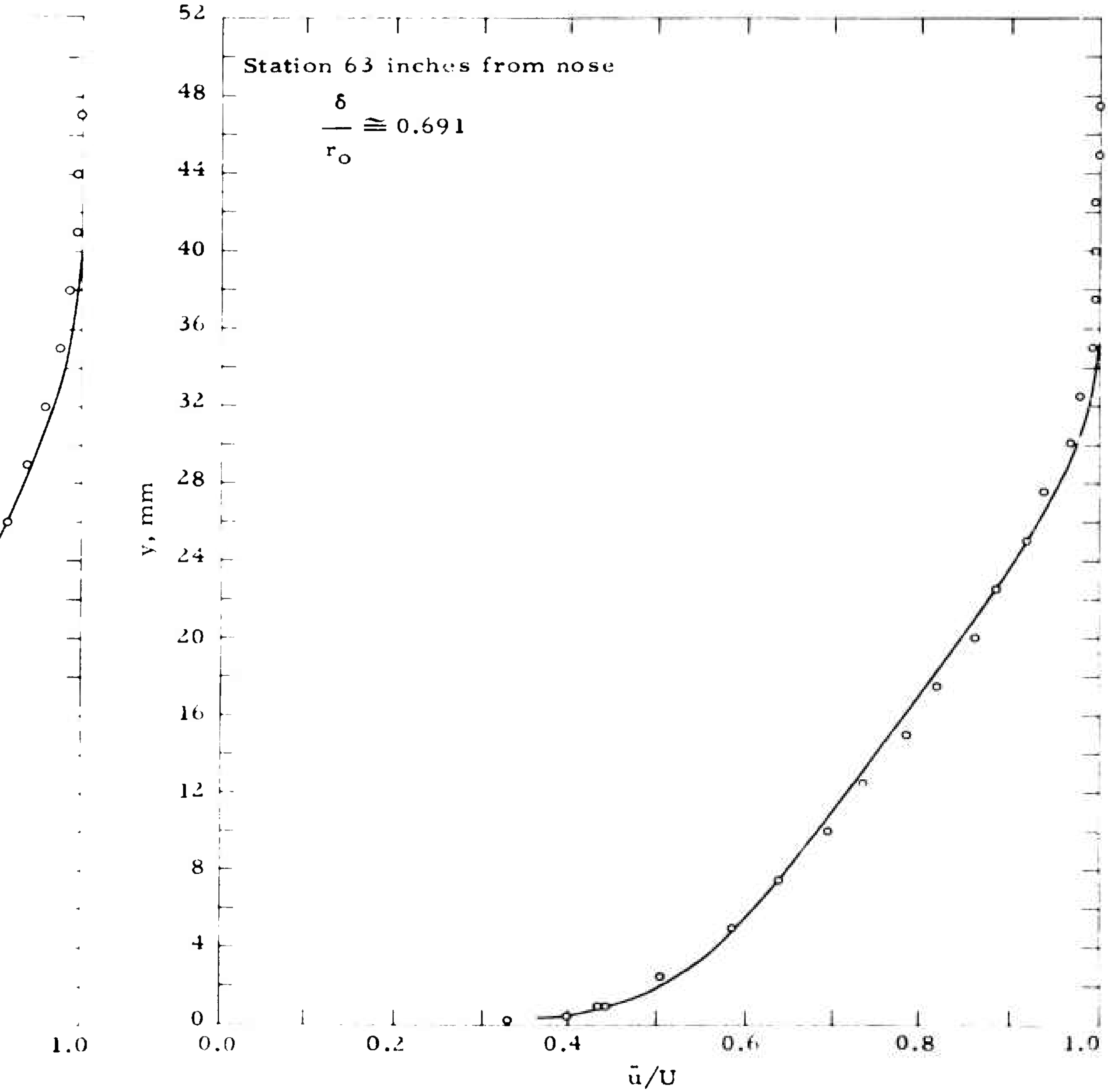
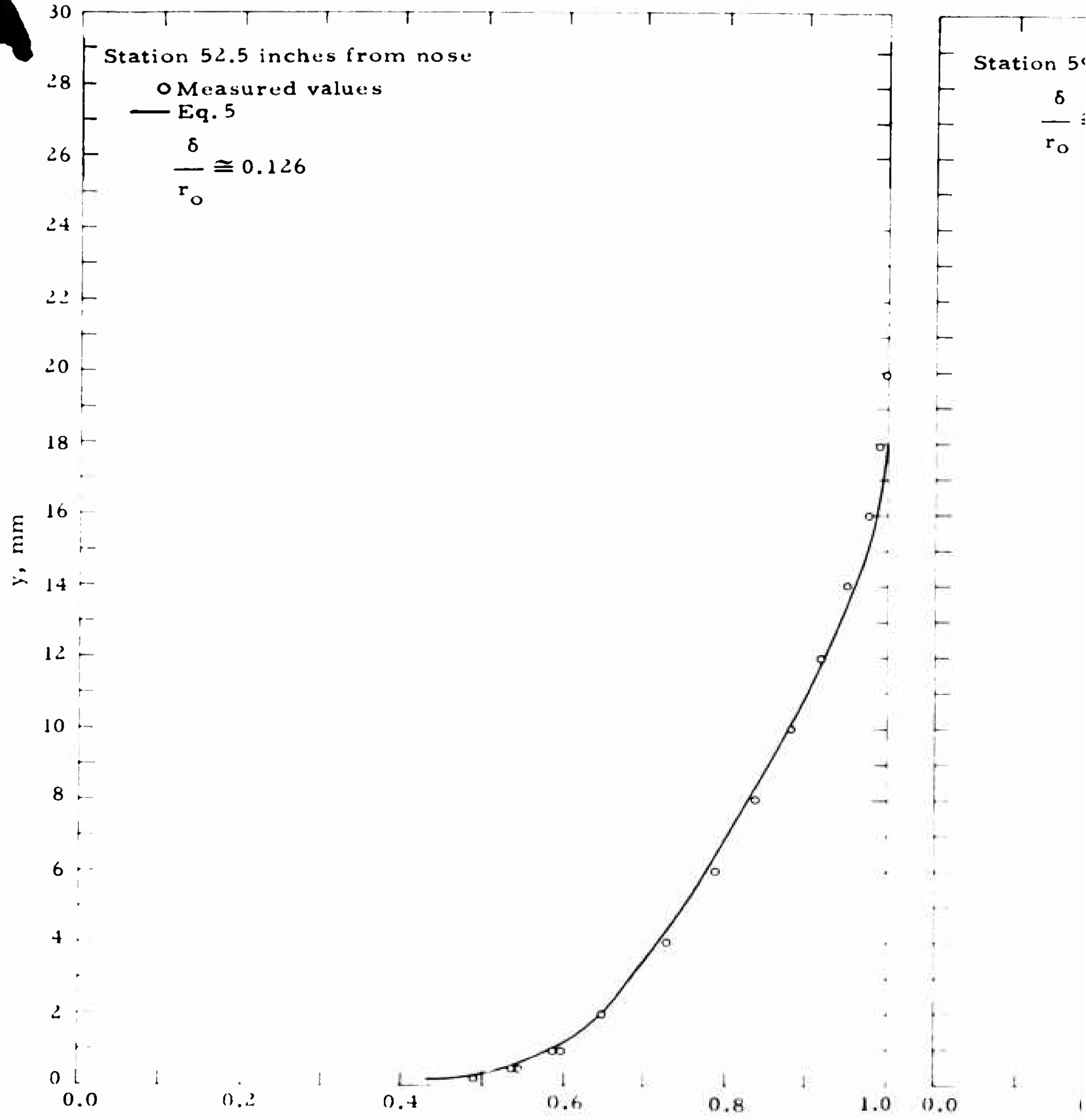
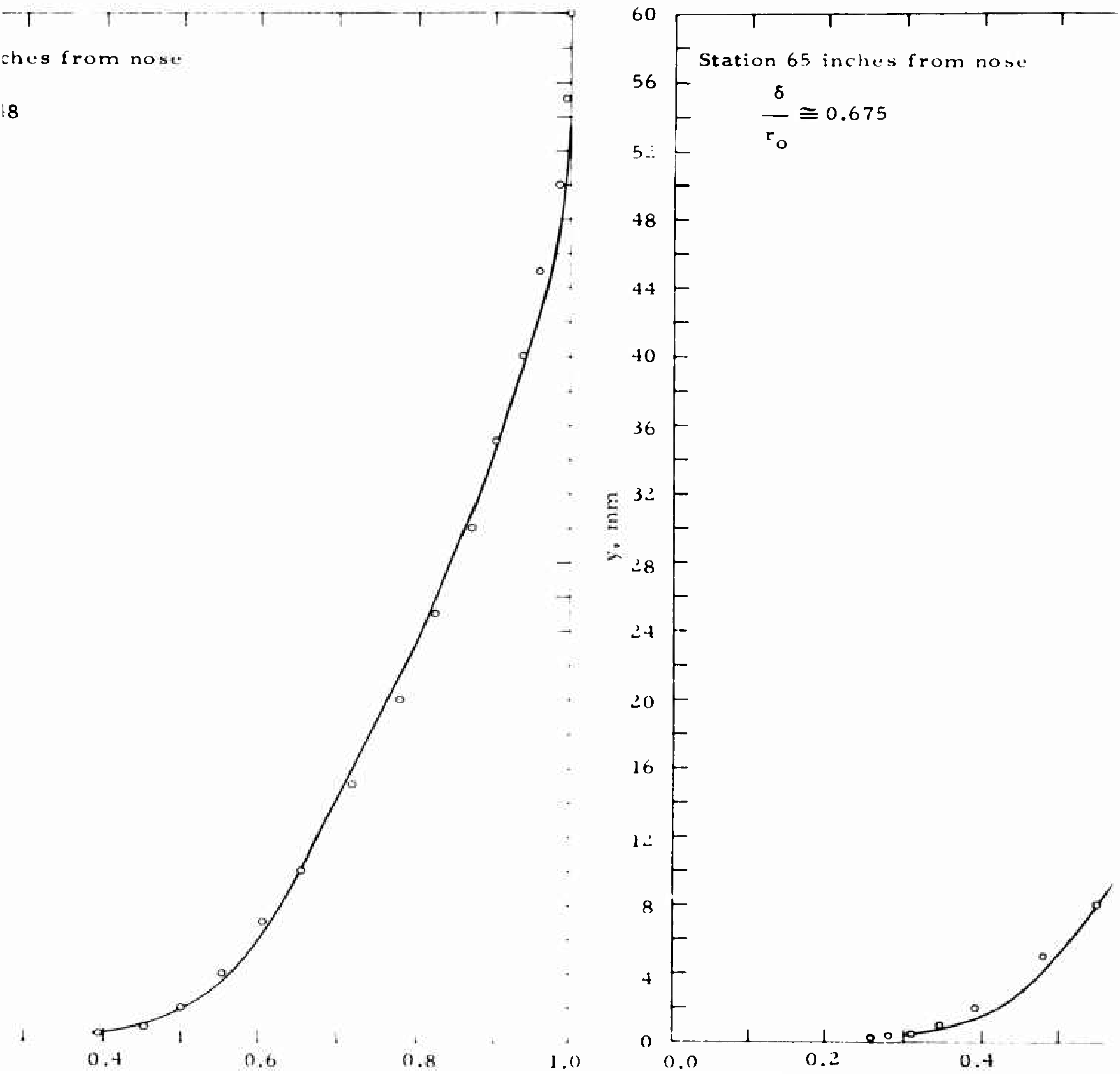


FIG. 1. Equation 5 Compared With Boundary-Layer Measurements Mad
(a) Lyon (Ref. 6), Body A.



y Several Investigators.


Station 52.5 inches from nose
Measured values
Eq. 5
$\frac{\delta}{r_o} \cong 0.126$
y, mm
30
28
26
24
22
20
18
16
14
12
10
8
6
4
2
0
0.0
0.2
0.4
0.6
0.8
1.0
Station 5
$\frac{\delta}{r_o}$
0.0


ches from nose
18
Station 65 inches from nose
$\frac{\delta}{r_o} \cong 0.675$
y, mm
60
56
52
48
44
40
36
32
28
24
20
16
12
8
4
0
0.4
0.6
0.8
1.0
0.0
0.2
0.4

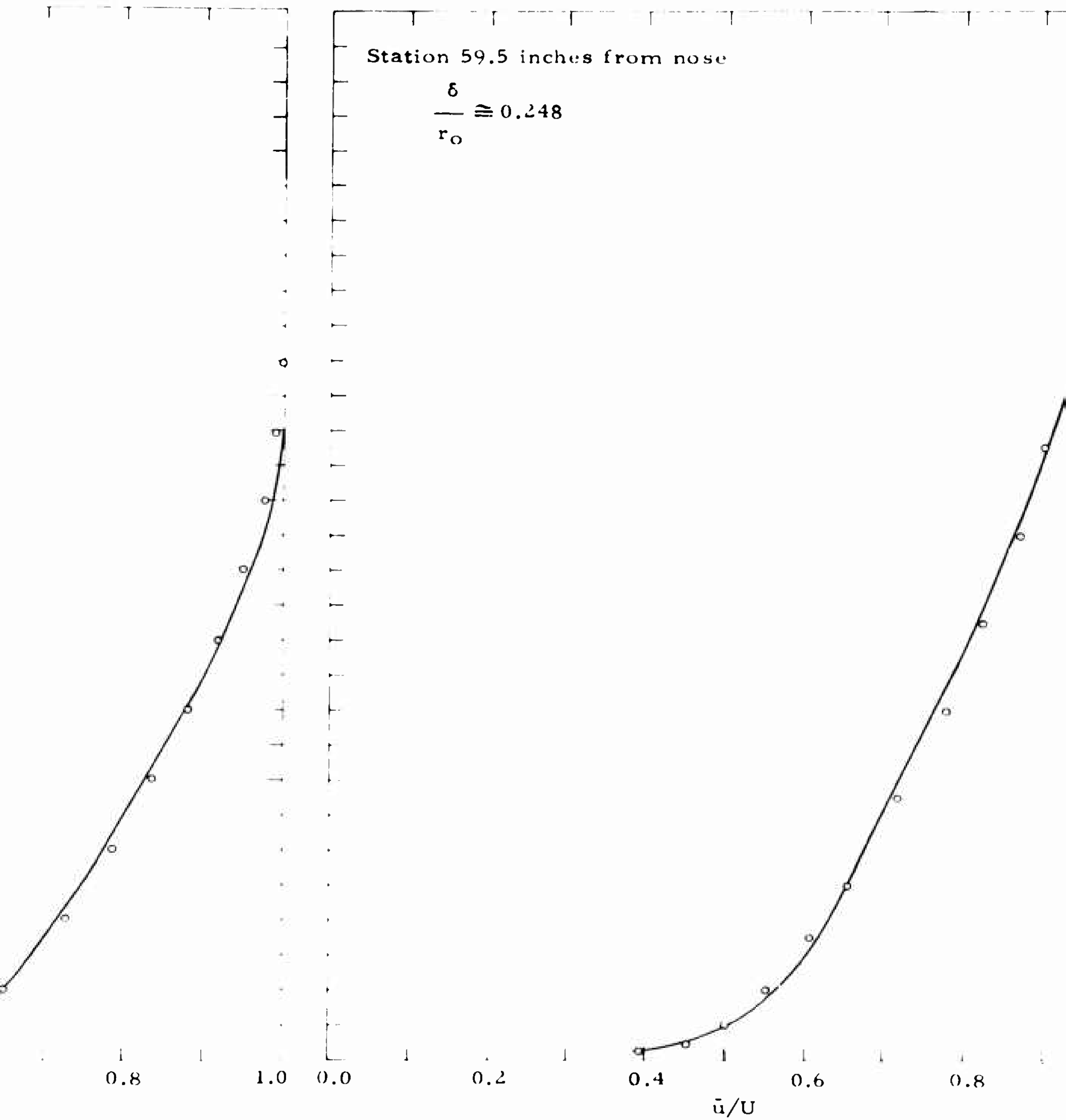
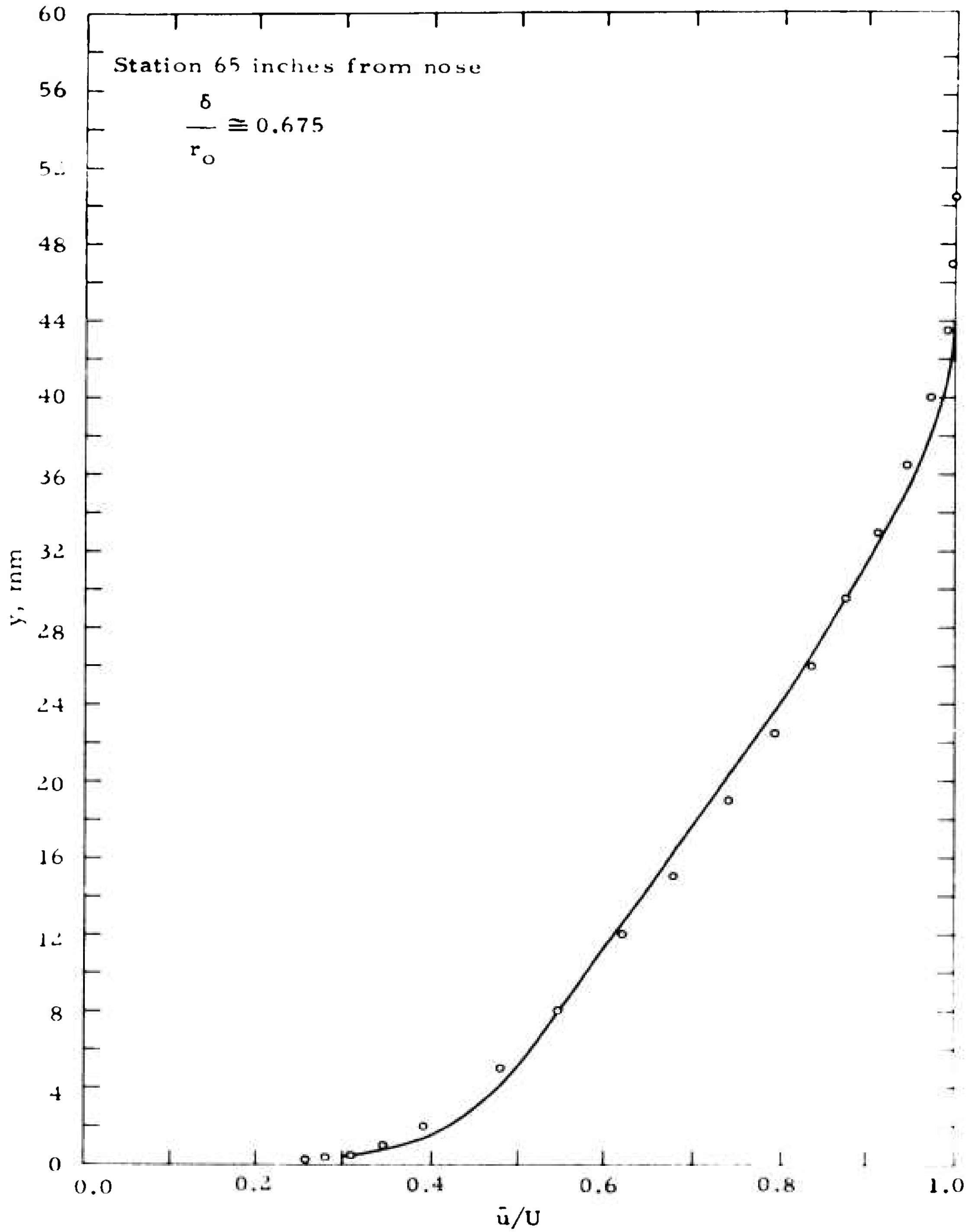


FIG. 1(b). Lyon (Ref. 6), Body B.


Station 65 inches from nose
$\frac{\delta}{r_o} \cong 0.675$
y, mm
ū/U
0.0
0.2
0.4
0.6
0.8
1.0
0
4
8
12
16
20
24
28
32
36
40
44
48
52
56
60

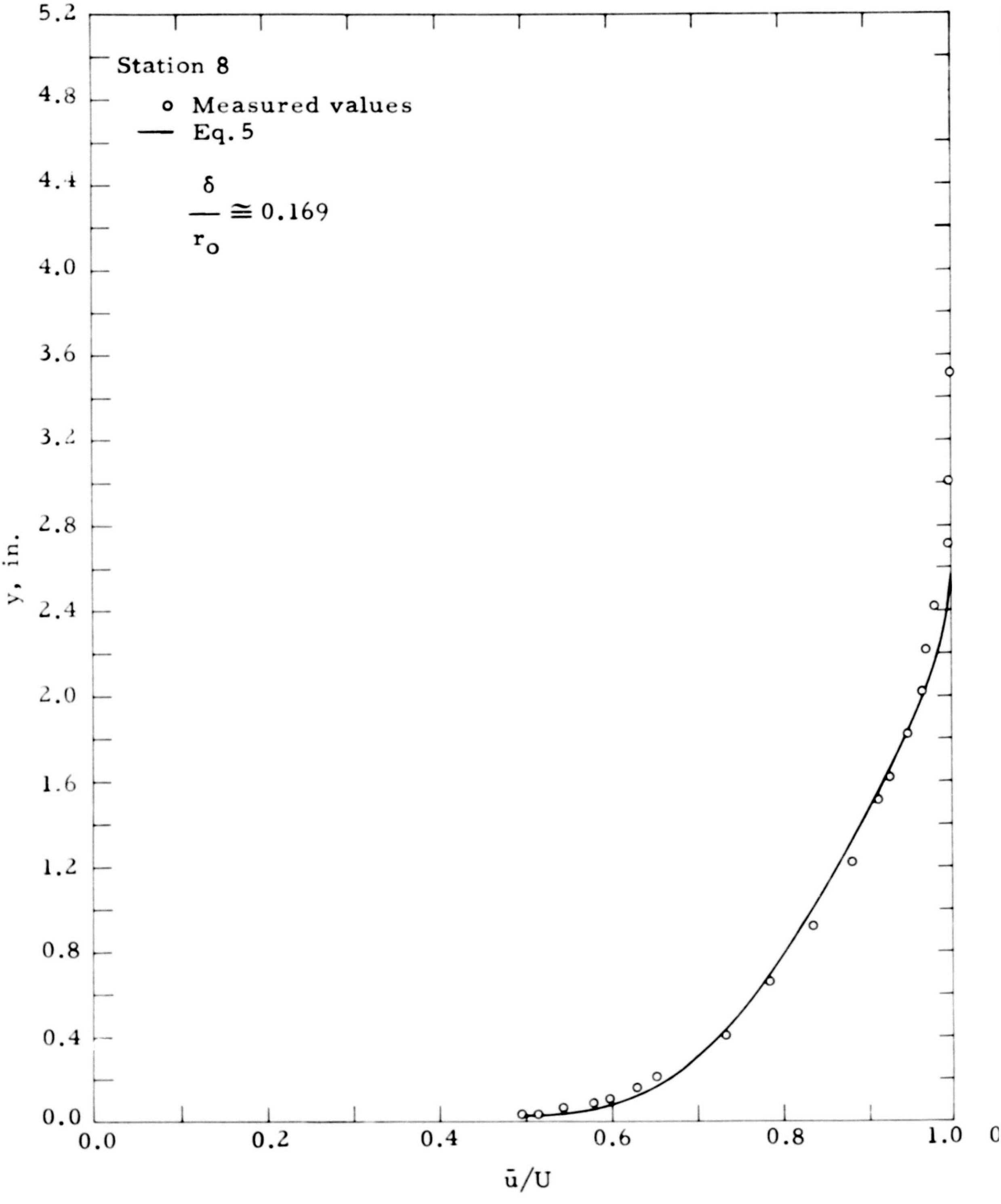
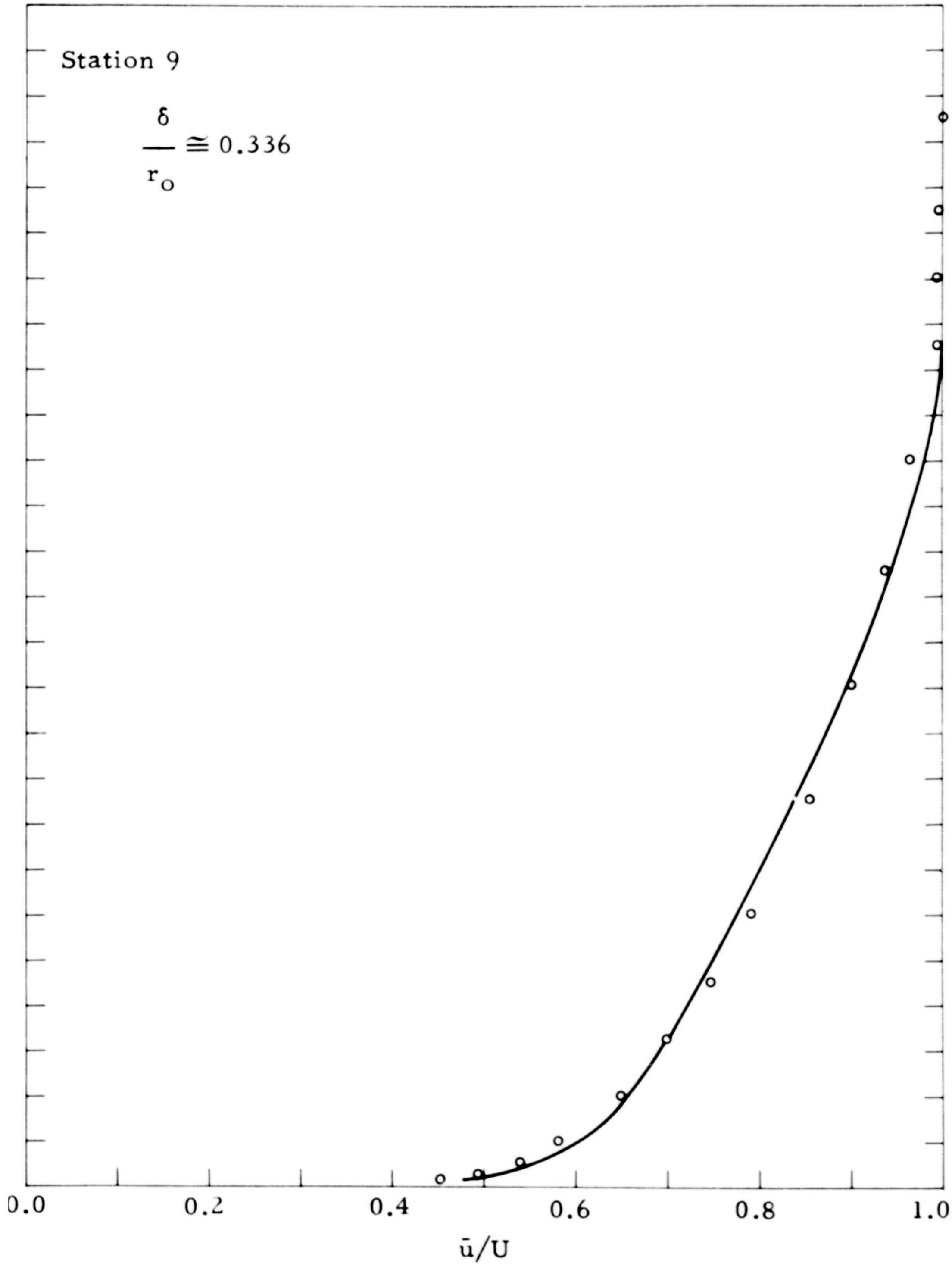
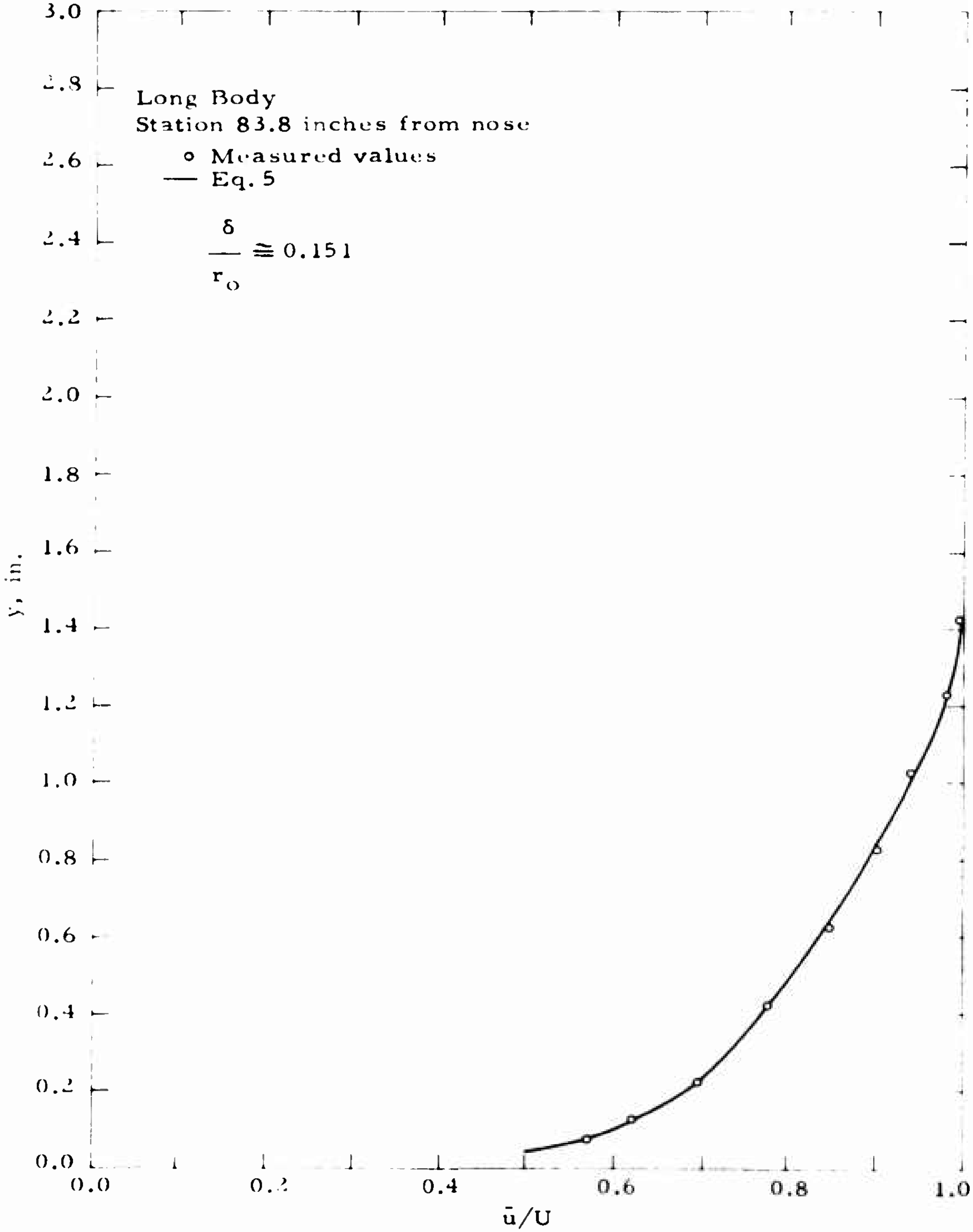


FIG. 1(c). Freeman (Ref. 7),



1/40-Scale Akron Airship.



3.0
2.8
2.6
2.4
2.2
2.0
1.8
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
y, in.
Long Body
Station 83.8 inches from nose
o Measured values
— Eq. 5
$\frac{\delta}{r_o} \cong 0.151$
0.0
0.2
0.4
0.6
0.8
1.0
$\bar{u}/U$

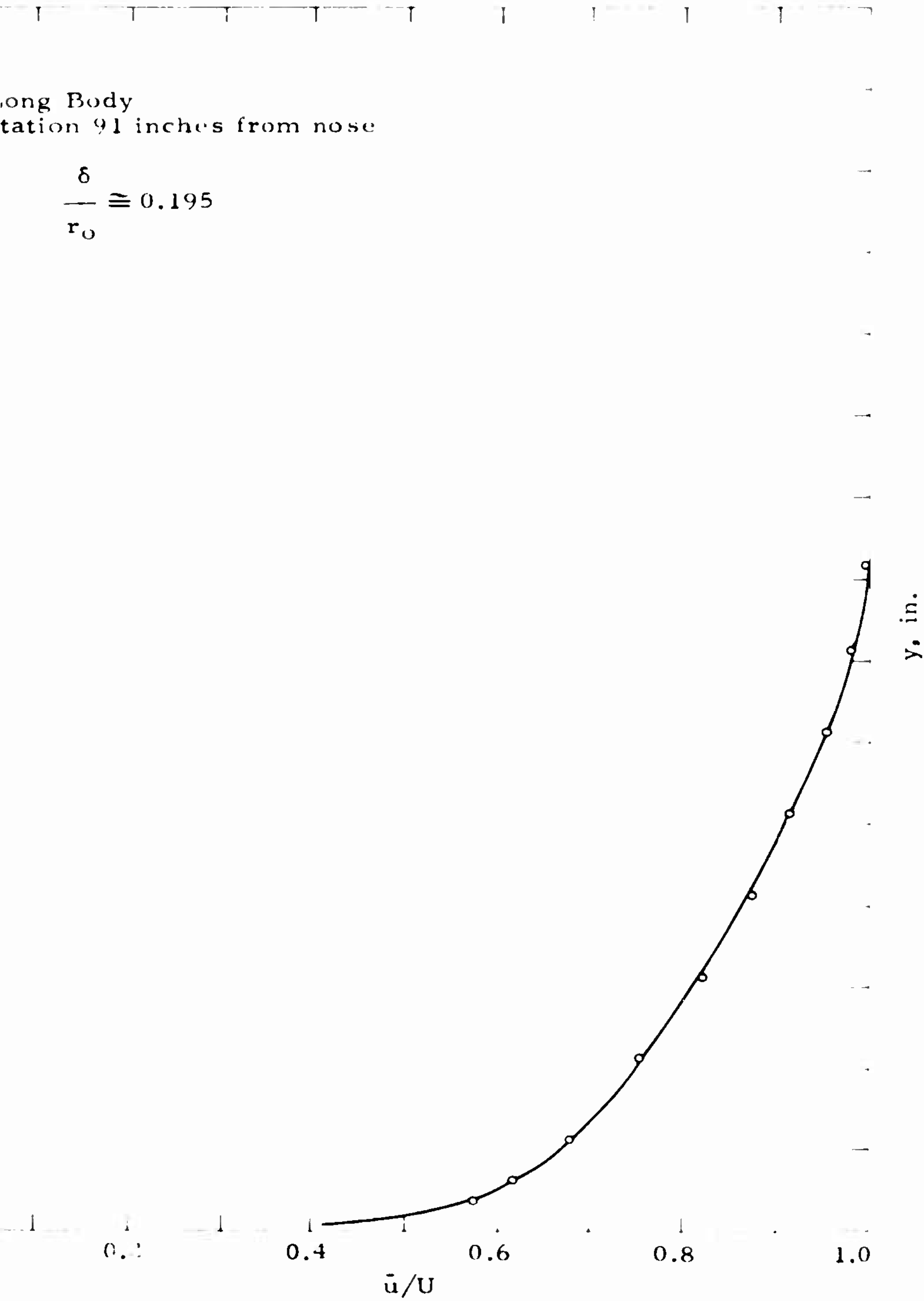
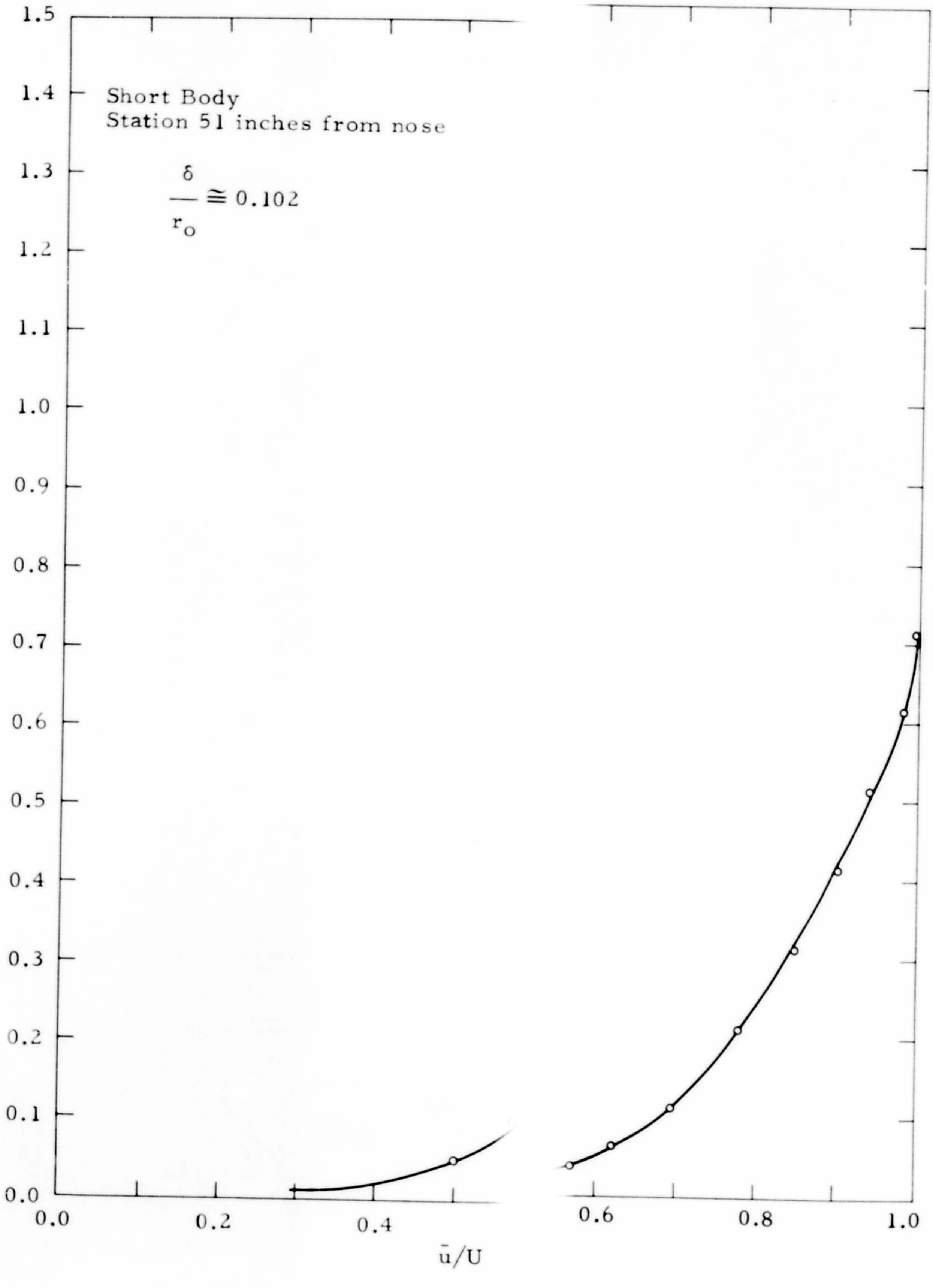


FIG. 1(d) Ower and Hutton (Ref. 8)



1.5
1.4
1.3
1.2
1.1
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
Short Body
Station 51 inches from nose
$\frac{\delta}{r_o} \cong 0.102$
0.0
0.2
0.4
0.6
0.8
1.0
$\bar{u}/U$

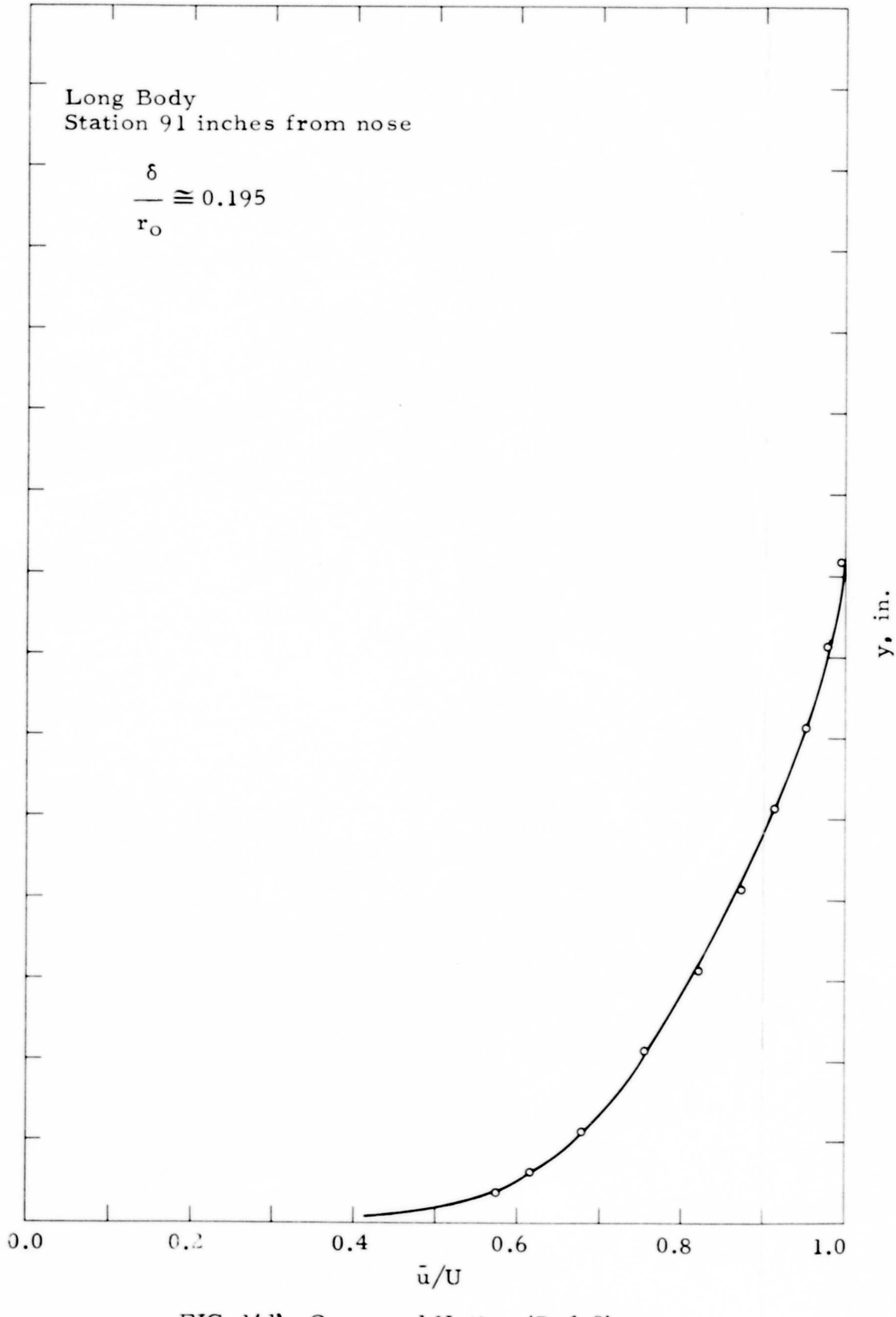
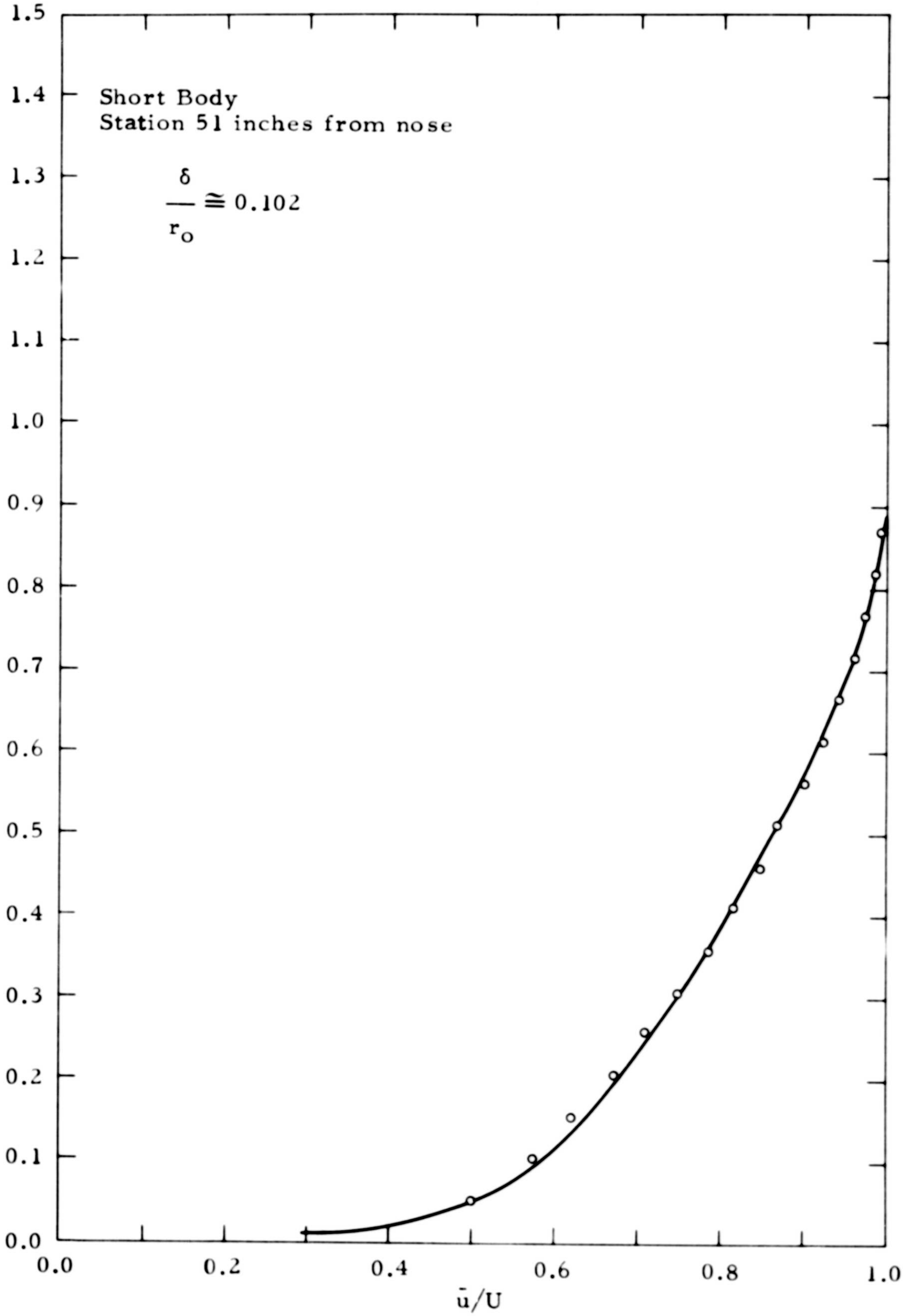


FIG. 1(d). Ower and Hutton (Ref. 8).


Short Body
Station 51 inches from nose
$\frac{\delta}{r_o} \cong 0.102$
1.5
1.4
1.3
1.2
1.1
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
0.0
0.2
0.4
0.6
0.8
1.0
$\bar{u}/U$

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AN ANALYTICAL APPROXIMATION TO EQUATION 5

Coles' expression for the mean velocity profiles, Eq. 5, has two drawbacks. First, it is not an analytical expression since $W(y/\delta)$ is given as a table of values so that manipulations involving integration must be carried out in part by numerical integration. Second, it does not satisfy the boundary condition at the edge of the boundary layer that $\partial\bar{u}/\partial y = 0$ when $y = \delta$. Although neither of these drawbacks seriously impedes its usefulness, it would be desirable to have a simple analytical expression amenable to straightforward integration that would closely approximate Eq. 5 and also satisfy the boundary condition $\partial\bar{u}/\partial y = 0$ when $y = \delta$. Such an expression was found in the form

$$\frac{\bar{u}}{v_*} = \frac{1}{\kappa} \ln \frac{y v_*}{\nu} + C + \frac{A}{\kappa}\left(\frac{y}{\delta}\right)^n + \frac{B}{\kappa}\left(\frac{y}{\delta}\right)^m \tag{12}$$

Applying the boundary condition $\partial\bar{u}/\partial y = 0$ when $y = \delta$, Eq. 12 becomes

$$\frac{\bar{u}}{v_*} = \frac{1}{\kappa} \ln \frac{y v_*}{\nu} + C + \frac{A}{\kappa}\left(\frac{y}{\delta}\right)^n - \frac{1}{\kappa}\left(\frac{1}{m} + \frac{n}{m} A\right)\left(\frac{y}{\delta}\right)^m \tag{13}$$

The profile parameter A in Eq. 13 plays the same role as the profile parameter Π in Eq. 5. By choosing $n = 2.50$ and $m = 2.75$ in Eq. 13, it very closely approximates Eq. 5. A comparison between Eq. 13 and Eq. 5 showing this fact can be made as follows.

If Eq. 13 and 5 are both fit to the same velocity profile, the boundary-layer thickness, δ, defined by Eq. 13 will not be exactly equal to that defined by Eq. 5. To avoid confusion, from this point on, a subscript c will be put on the boundary-layer thickness defined by Coles' Eq. 5 to differentiate it from the boundary-layer thickness defined by Eq. 13.

Applying the boundary conditon $\bar{u} = U$ when $y = \delta$ to Eq. 13 and subtracting Eq. 13 from this result yields

$$\frac{U - \bar{u}}{v_*} = -\frac{1}{\kappa} \ln \frac{y}{\delta} + \frac{A}{\kappa}\left[1 - \left(\frac{y}{\delta}\right)^n\right] - \frac{1}{\kappa}\left(\frac{1}{m} + \frac{n}{m} A\right)\left[1 - \left(\frac{y}{\delta}\right)^m\right] \tag{14}$$

Substituting Eq. 10 into Eq. 6 and 7 and also Eq. 14 into Eq. 6 and 7, and carrying out the integrations,[5] there is obtaired

---

[5] The same comments made in footnote 4 concerning Eq. 5 and 10 apply to Eq. 13 and 14.

$$\Delta = \delta_c \frac{1 + \Pi}{\kappa}$$

$$G = \frac{2 + 3.200\Pi + 1.522\Pi^2}{\kappa(1 + \Pi)}$$

$$\left.\begin{aligned} \Delta &= \delta \frac{0.7333 + 0.04762A}{\kappa} \\ G &= \frac{1.406 + 0.1162A + 0.003297A^2}{\kappa(0.7333 + 0.04762A)} \end{aligned}\right\} \quad \begin{aligned} n &= 2.50 \\ m &= 2.75 \end{aligned}$$

Specifying that both profiles have the same values of Δ and G, the equations above allow the determination of $A = f(\Pi)$ and $\delta/\Delta = f(\delta_c/\Delta) = f(\Pi)$. With these relationships known, a direct comparison of Eq. 5 and 13 can be made. In order to make this comparison independent of the value of v_*, Eq. 5 and 13 are rewritten in the following form.

$$\frac{\bar{u}}{v_*} - \frac{1}{\kappa} \ln \frac{\Delta v_*}{\nu} - C = \frac{1}{\kappa} \ln \frac{y}{\Delta} + \frac{\Pi}{\kappa} W\left(\frac{y}{\delta_c}\right) \tag{15}$$

$$\frac{\bar{u}}{v_*} - \frac{1}{\kappa} \ln \frac{\Delta v_*}{\nu} - C = \frac{1}{\kappa} \ln \frac{y}{\Delta} + \frac{A}{\kappa}\left(\frac{y}{\delta}\right)^n - \frac{1}{\kappa}\left(\frac{1}{m} + \frac{n}{m} A\right)\left(\frac{y}{\delta}\right)^m \tag{16}$$

The comparison between Eq. 15 and 16 is shown in Fig. 2a - g for values of Π ranging from 0.1 to 5.0, a range that covers the values most commonly encountered in practice. The agreement is excellent, making it reasonable to use Eq. 13 and 14 in place of Eq. 5 and 10 in all subsequent analyses. Although it is understood that n = 2.50 and m = 2.75, the symbols n and m are retained in the following work to simplify the writing of expressions.

COORDINATE SYSTEM

The most convenient coordinate system for analyzing the axisymmetric flow over a body of revolution is the orthogonal curvilinear system shown in Fig. 3. Here, x is the distance measured from the nose along the surface of the body in a meridian plane; y is the distance from the surface of the body measured along an outward normal to the body surface; γ is the angular position of points around the body; K denotes the curvature of the body surface in the meridian plane; and β denotes the angle between a tangent to the body surface and the axis

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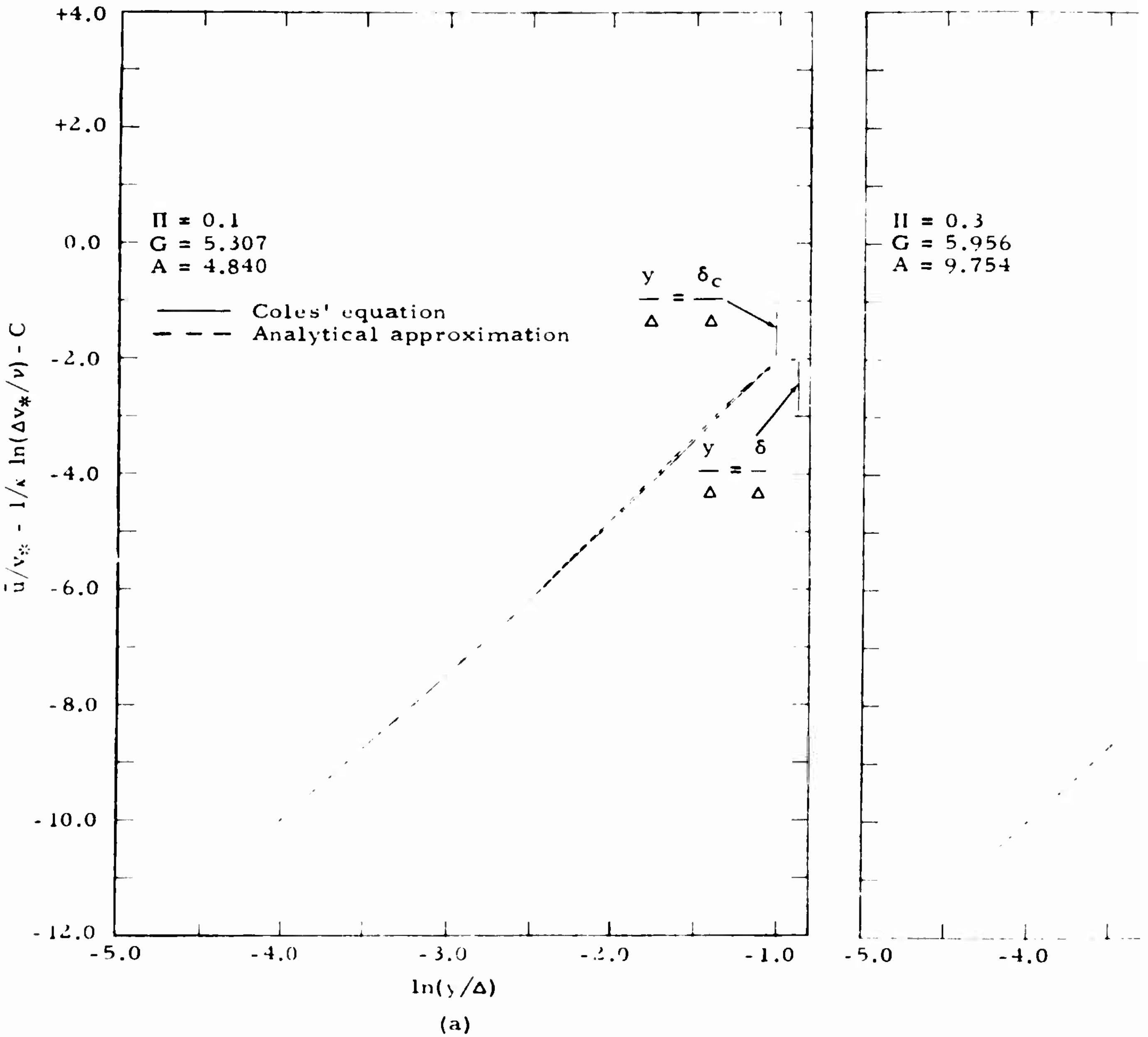
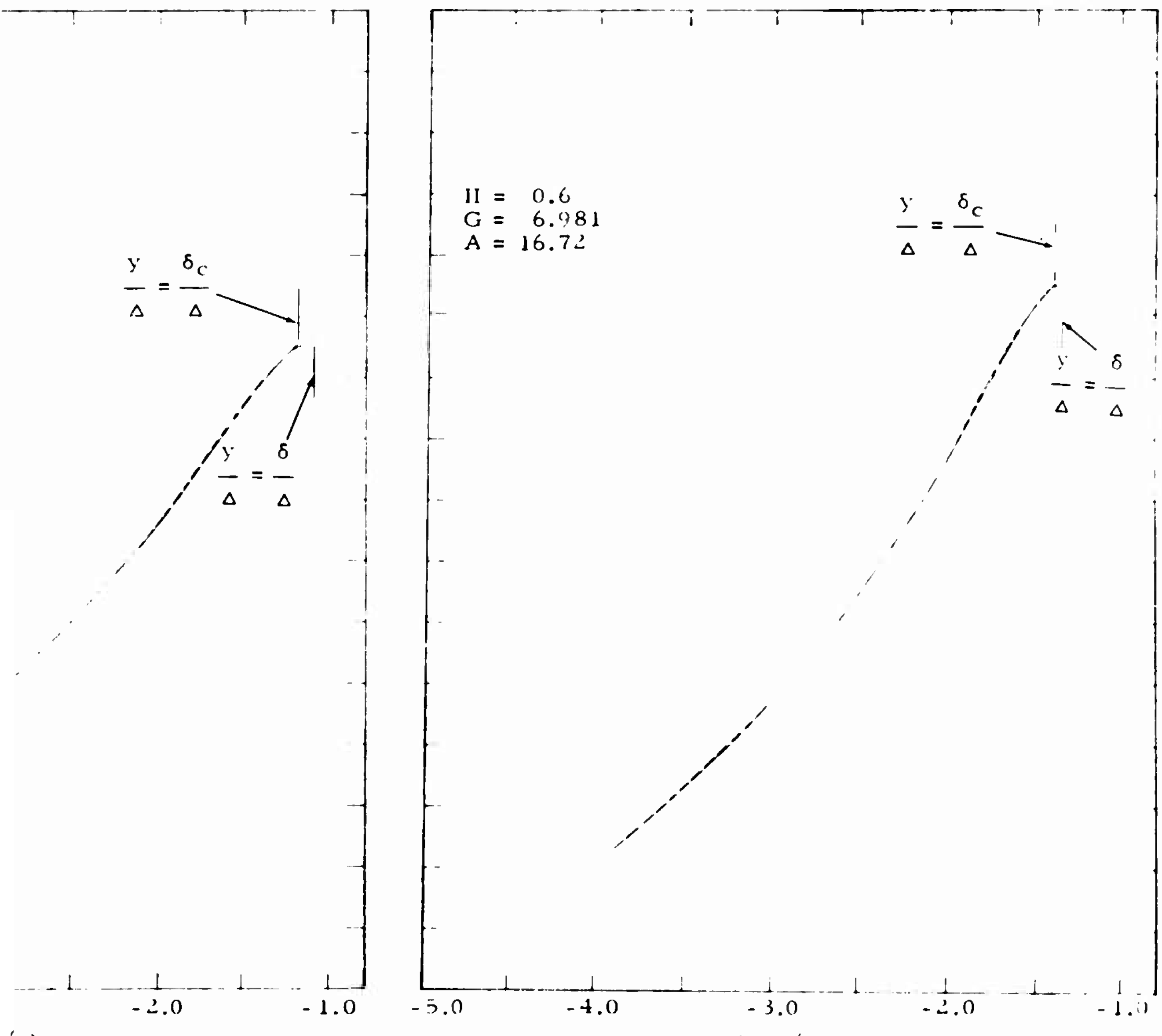
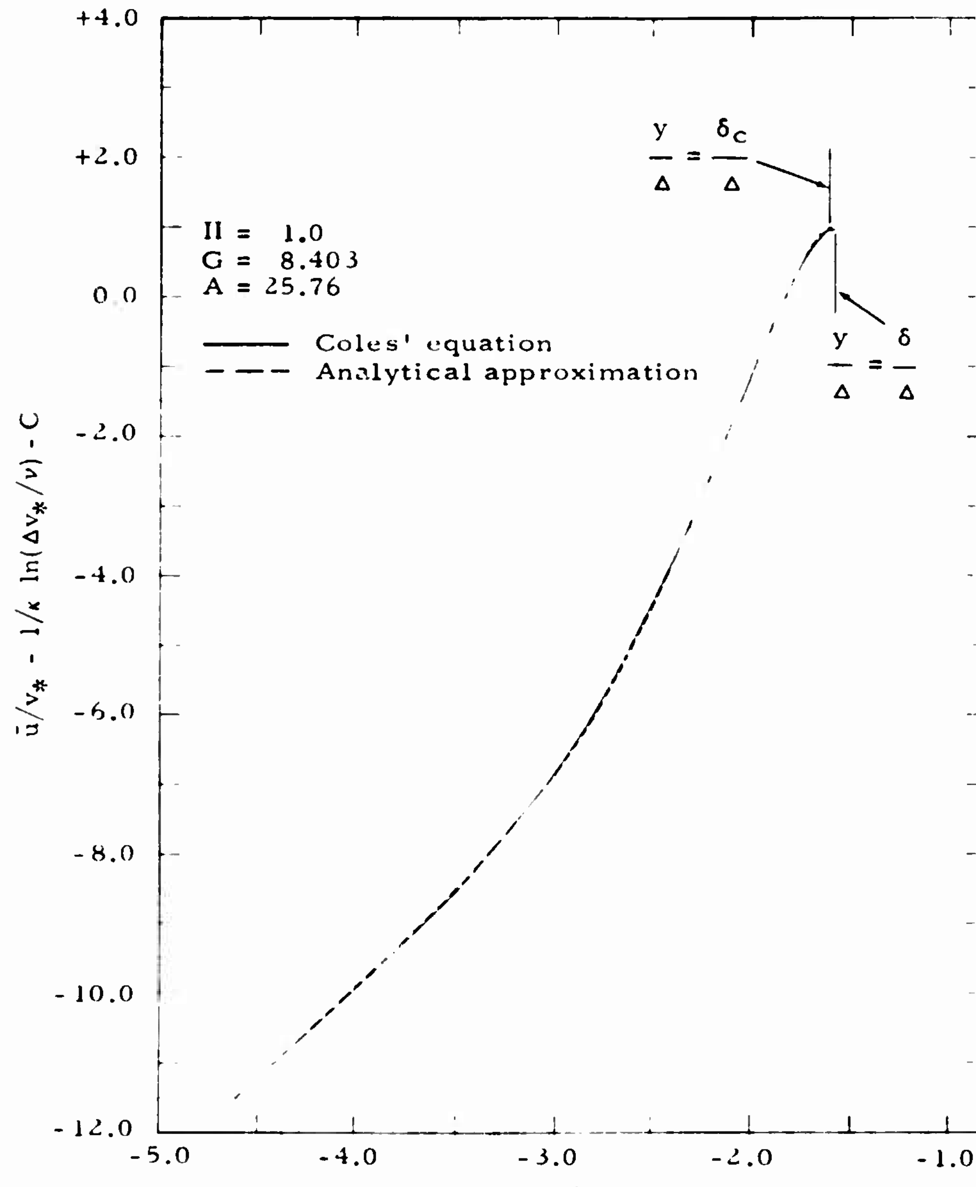


FIG. 2. Comparison of Coles' Equation for



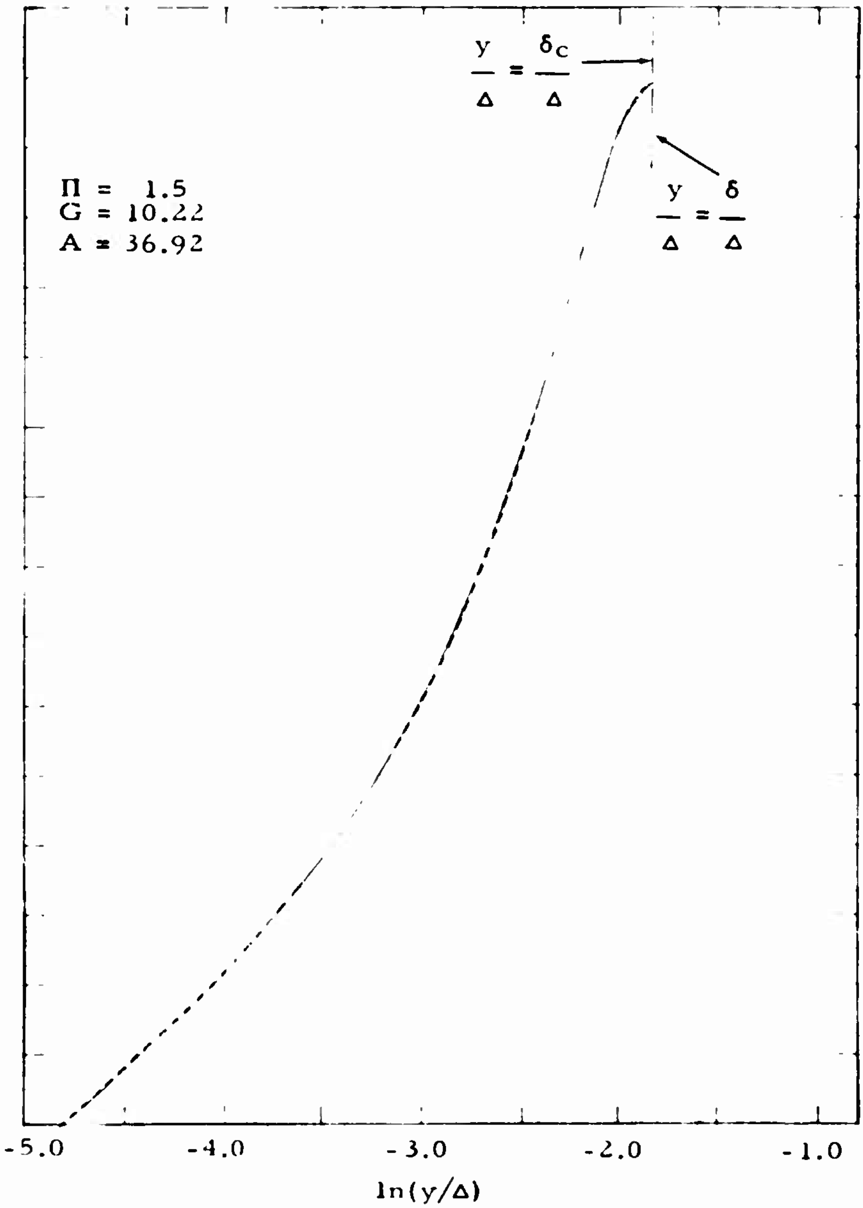
(c)

Velocity Profiles With Analytical Approximation.


+4.0
+2.0
0.0
-2.0
-4.0
-6.0
-8.0
-10.0
-12.0
$\bar{u}/v_* - 1/\kappa \ln(\Delta v_*/\nu) - C$
$\Pi = 1.0$
$G = 8.403$
$A = 25.76$
Coles' equation
Analytical approximation
$\frac{y}{\Delta} = \frac{\delta_c}{\Delta}$
$\frac{y}{\Delta} = \frac{\delta}{\Delta}$
-5.0
-4.0
-3.0
-2.0
-1.0
$\ln(y/\Delta)$

(d)

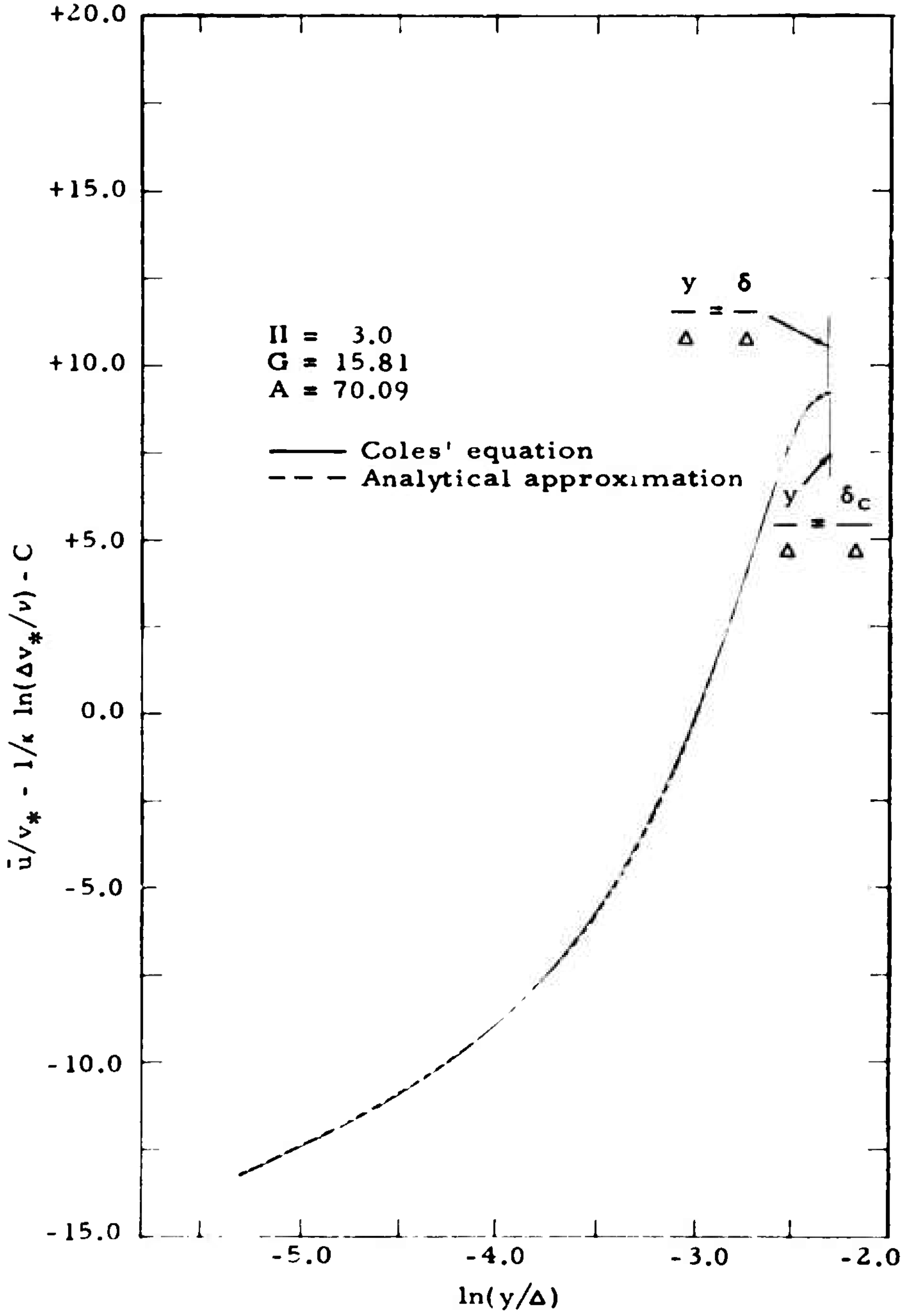
FIG.



$\frac{y}{\Delta} = \frac{\delta_c}{\Delta}$
$\frac{y}{\Delta} = \frac{\delta}{\Delta}$
$\Pi = 1.5$
$G = 10.22$
$A = 36.92$
-5.0
-4.0
-3.0
-2.0
-1.0
$\ln(y/\Delta)$

(e)

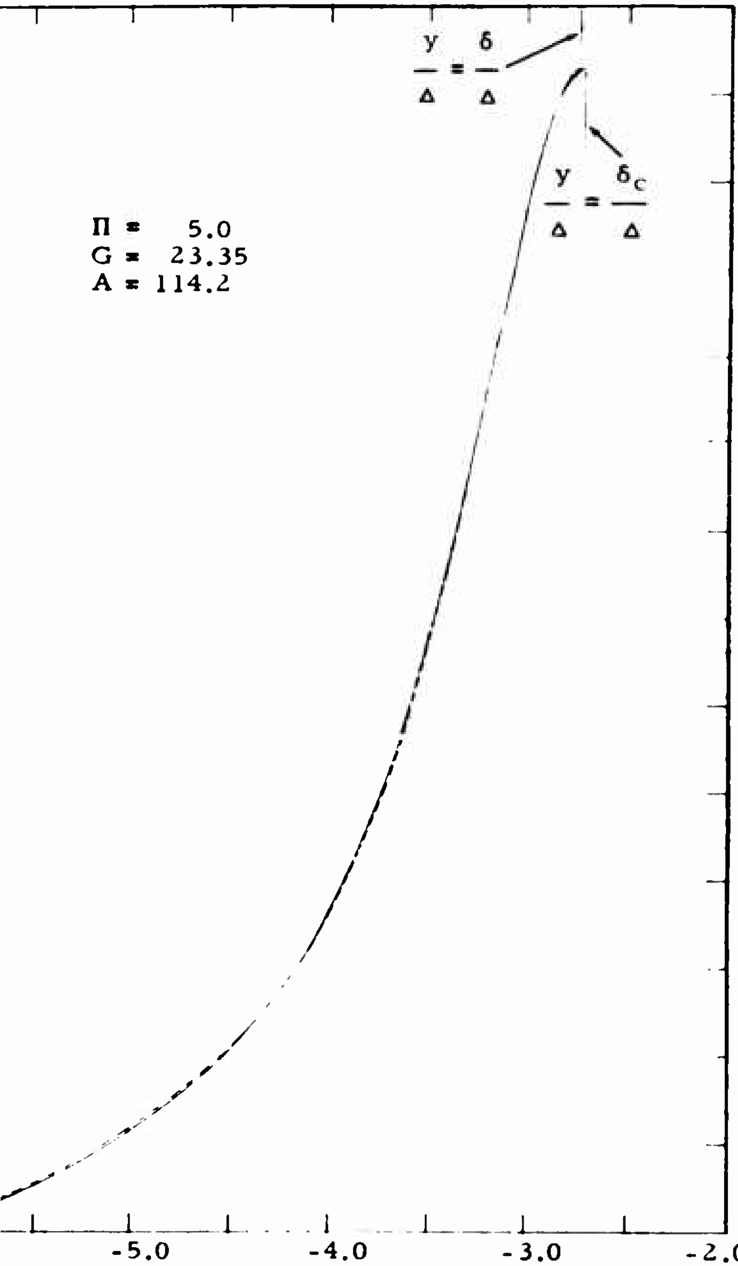
(Contd.)



+20.0
+15.0
+10.0
+5.0
0.0
-5.0
-10.0
-15.0
$\bar{u}/v_* - 1/\kappa\ \ln(\Delta v_*/\nu) - C$
Π = 3.0
G = 15.81
A = 70.09
Coles' equation
Analytical approximation
$\frac{y}{\Delta} = \frac{\delta}{\Delta}$
$\frac{y}{\Delta} = \frac{\delta_c}{\Delta}$
-5.0
-4.0
-3.0
-2.0
$\ln(y/\Delta)$

(f)

FIG. 2. (C



$\frac{y}{\Delta} = \frac{\delta}{\Delta}$
$\frac{y}{\Delta} = \frac{\delta_c}{\Delta}$
Π = 5.0
G = 23.35
A = 114.2
-5.0
-4.0
-3.0
-2.0
ln(y/Δ)

(g)

ntd.)

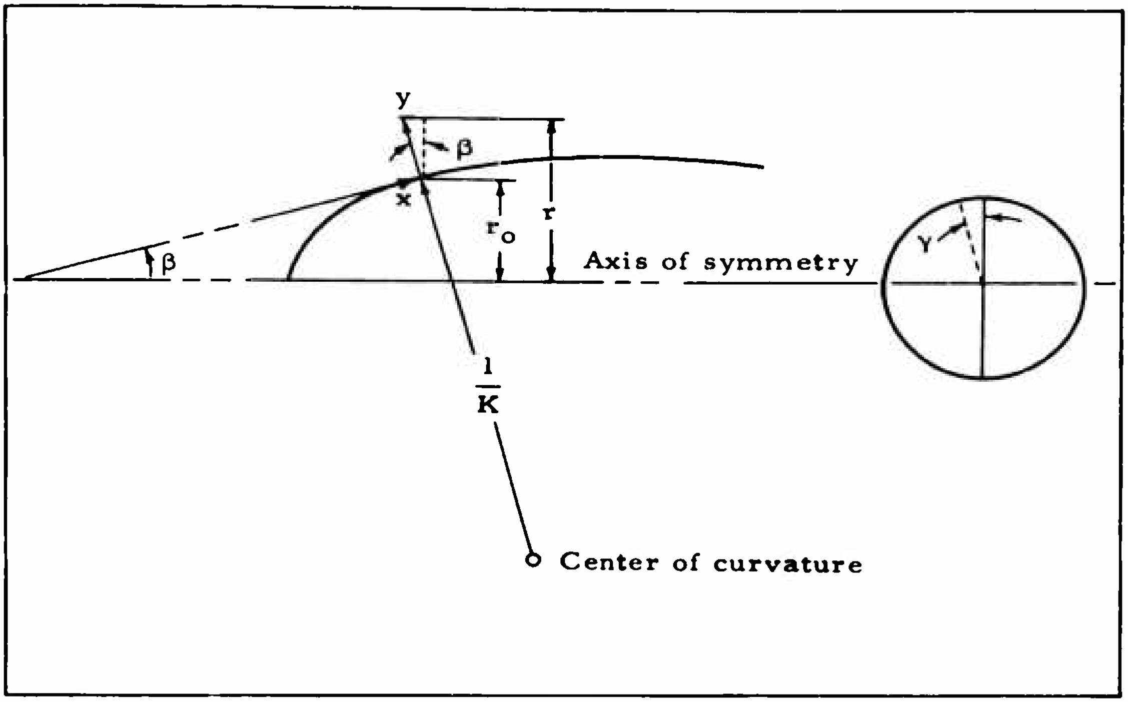


FIG. 3. Curvilinear Coordinate System Used in Analysis.

of symmetry. The scale factors[6] for this curvilinear system corresponding to the coordinates x, y, and γ are

$$
\begin{aligned}
h_x &= 1 + Ky \\
h_y &= 1 \\
h_\gamma &= r = r_o + y \cos\beta
\end{aligned}
\tag{17}
$$

VECTOR OPERATIONS

In an orthogonal curvilinear coordinate system, the operations involving the del, ∇, operator take on a form different from that encountered in a rectangular system.[6] For an orthogonal curvilinear system with coordinates x_1, x_2, x_3 and scale factors h_1, h_2, h_3, the following expressions are valid.

---

[6] A discussion of curvilinear coordinates may be found in Ref. 9.

$$\nabla\phi = \text{grad}\,\phi = \frac{1}{h_1}\frac{\partial\phi}{\partial x_1}e_1 + \frac{1}{h_2}\frac{\partial\phi}{\partial x_2}e_2 + \frac{1}{h_3}\frac{\partial\phi}{\partial x_3}e_3$$

$$\nabla\cdot\vec{B} = \text{div}\,\vec{B} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x_1}(h_2h_3B_1) + \frac{\partial}{\partial x_2}(h_3h_1B_2) + \frac{\partial}{\partial x_3}(h_1h_2B_3)\right] \tag{18}$$

$$\nabla\times\vec{B} = \text{curl}\,\vec{B} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1e_1 & h_2e_2 & h_3e_3 \\ \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_3} \\ h_1B_1 & h_2B_2 & h_3B_3 \end{vmatrix}$$

In the above, ϕ is any scalar and $\vec{B}$ is any vector having components B_1, B_2, B_3 in the x_1, x_2, x_3 system, and e_1, e_2, and e_3 are the unit vectors in the directions of increasing x_1, x_2, and x_3, respectively. It is of course understood that the x_1, x_2, x_3 system is right-handed.

THE NAVIER-STOKES AND CONTINUITY EQUATIONS

The equations governing the flow of an incompressible, viscous Newtonian fluid are the Navier-Stokes and continuity equations. For this incompressible case, and neglecting gravitational forces, these equations may be written (Ref. 10) as follows.

$$\begin{gathered} \frac{\partial\vec{V}}{\partial t} - \vec{V}\times(\nabla\times\vec{V}) + \frac{1}{2}\nabla(\vec{V}\cdot\vec{V}) = -\frac{1}{\rho}\nabla p - \nu\nabla\times(\nabla\times\vec{V}) \\ \nabla\cdot\vec{V} = 0 \end{gathered} \tag{19}$$

In these equations $\vec{V}$ is the vector velocity of the fluid and p is the static pressure. The components of $\vec{V}$ in the directions of increasing x, y, and γ will be denoted as u, v, and w, respectively.

Expanding Eq. 19 using Eq. 17, and using Eq. 18 with

$$\begin{aligned} x_1 &= x & h_1 &= h_x \\ x_2 &= y & h_2 &= h_y \\ x_3 &= \gamma & h_3 &= h_\gamma \end{aligned}$$

there is obtained

(Flow along body)

$$
\begin{aligned}
&\frac{\partial u}{\partial t} + \frac{u}{(1+Ky)}\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{uv}{(1+Ky)}K + \frac{w}{r}\frac{\partial u}{\partial \gamma} - \frac{w^2}{r(1+Ky)}\frac{\partial r}{\partial x} \\
&\quad = -\frac{1}{\rho(1+Ky)}\frac{\partial p}{\partial x} - \nu\Bigg[\frac{1}{(1+Ky)}\frac{\partial^2 v}{\partial x \partial y} + \frac{\cos\beta}{r(1+Ky)}\frac{\partial v}{\partial x} \\
&\qquad - \frac{K}{(1+Ky)^2}\frac{\partial v}{\partial x} - \frac{K}{(1+Ky)}\frac{\partial u}{\partial y} - \frac{K\cos\beta}{r(1+Ky)}u + \frac{K^2}{(1+Ky)^2}u \\
&\qquad - \frac{\partial^2 u}{\partial y^2} - \frac{\cos\beta}{r}\frac{\partial u}{\partial y} - \frac{1}{r^2}\frac{\partial^2 u}{\partial \gamma^2} + \frac{1}{r^2(1+Ky)}\frac{\partial r}{\partial x}\frac{\partial w}{\partial \gamma} + \frac{1}{r(1+Ky)}\frac{\partial^2 w}{\partial x \partial \gamma}\Bigg]
\end{aligned}
$$

(Flow normal to body)

$$
\begin{aligned}
&\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial y} + \frac{u}{(1+Ky)}\frac{\partial v}{\partial x} + \frac{w}{r}\frac{\partial v}{\partial \gamma} - \frac{w^2}{r}\cos\beta - \frac{u^2}{(1+Ky)}K \\
&\quad = -\frac{1}{\rho}\frac{\partial p}{\partial y} - \nu\Bigg[\frac{\cos\beta}{r^2}\frac{\partial w}{\partial \gamma} + \frac{1}{r}\frac{\partial^2 w}{\partial y \partial \gamma} - \frac{1}{r^2}\frac{\partial^2 v}{\partial \gamma^2} - \frac{1}{(1+Ky)^2}\frac{\partial^2 v}{\partial x^2} \\
&\qquad - \frac{1}{r(1+Ky)^2}\frac{\partial r}{\partial x}\frac{\partial v}{\partial x} + \frac{y}{(1+Ky)^3}\frac{\partial K}{\partial x}\frac{\partial v}{\partial x} + \frac{K}{(1+Ky)^2}\frac{\partial u}{\partial x} + \frac{u}{(1+Ky)^2}\frac{\partial K}{\partial x} \\
&\qquad + \frac{uK}{r(1+Ky)^2}\frac{\partial r}{\partial x} - \frac{uKy}{(1+Ky)^3}\frac{\partial K}{\partial x} + \frac{1}{(1+Ky)}\frac{\partial^2 u}{\partial y \partial x} + \frac{1}{r(1+Ky)}\frac{\partial r}{\partial x}\frac{\partial u}{\partial y}\Bigg]
\end{aligned}
$$

(Flow around body)

$$
\begin{aligned}
&\frac{\partial w}{\partial t} + \frac{w}{r}\frac{\partial w}{\partial \gamma} + \frac{u}{(1+Ky)}\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + \frac{uw}{r(1+Ky)}\frac{\partial r}{\partial x} + \frac{vw}{r}\cos\beta \\
&\quad = -\frac{1}{\rho r}\frac{\partial p}{\partial \gamma} - \nu\Bigg[\frac{1}{r(1+Ky)}\frac{\partial^2 u}{\partial \gamma \partial x} - \frac{1}{r^2(1+Ky)}\frac{\partial r}{\partial x}\frac{\partial u}{\partial \gamma} - \frac{w}{r(1+Ky)^2}\frac{\partial^2 r}{\partial x^2} \\
&\qquad - \frac{1}{r(1+Ky)^2}\frac{\partial r}{\partial x}\frac{\partial w}{\partial x} + \frac{wy}{r(1+Ky)^3}\frac{\partial K}{\partial x}\frac{\partial r}{\partial x} + \frac{w}{r^2(1+Ky)^2}\frac{\partial r}{\partial x}
\end{aligned}
$$

$$- \frac{1}{(1 + Ky)^2} \frac{\partial^2 w}{\partial x^2} + \frac{y}{(1 + Ky)^3} \frac{\partial K}{\partial x} \frac{\partial w}{\partial x} - \frac{\cos\beta}{r} \frac{\partial w}{\partial y}$$

$$- \frac{K \cos\beta}{r(1 + Ky)} w + \frac{\cos^2\beta}{r^2} w - \frac{\partial^2 w}{\partial y^2} - \frac{K}{(1 + Ky)} \frac{\partial w}{\partial y}$$

$$\left. + \frac{1}{r} \frac{\partial^2 v}{\partial \gamma \partial y} + \frac{K}{r(1 + Ky)} \frac{\partial v}{\partial \gamma} - \frac{\cos\beta}{r^2} \frac{\partial v}{\partial \gamma} \right]$$

(Continuity)

$$\frac{\partial u}{\partial x} + \frac{u}{r} \frac{\partial r}{\partial x} + (1 + Ky) \frac{\partial v}{\partial y} + vK + \frac{(1 + Ky)\cos\beta}{r} v + \frac{(1 + Ky)}{r} \frac{\partial w}{\partial \gamma} = 0 \quad (20)$$

REYNOLDS' EQUATIONS

Applying Reynolds' procedure (Ref. 11) to Eq. 20, the velocity components and pressure for the turbulent flow are considered as being made up of a mean component designated by a bar and a fluctuation component designated by a prime.

$$u = \bar{u} + u' \qquad v = \bar{v} + v' \qquad w = \bar{w} + w' \qquad p = \bar{p} + p' \quad (21)$$

A mean component is determined by a time-average at a fixed point in space, so that

$$\overline{u'} = 0 \qquad \overline{v'} = 0 \qquad \overline{w'} = 0 \qquad \overline{p'} = 0 \quad (22)$$

Letting f and g be any two dependent variables and z be any independent variable, the following rules for operating on a mean time-average apply.

$$\bar{\bar{f}} = \bar{f} \qquad \overline{f + g} = \bar{f} + \bar{g}$$
$$\overline{\bar{f}g} = \bar{f}\bar{g} \qquad \overline{\frac{\partial f}{\partial z}} = \frac{\partial \bar{f}}{\partial z} \quad (23)$$

Restricting this development to the case of steady-state flow, and since the flow is axisymmetric, the following relationships hold.

$$\frac{\partial \bar{f}}{\partial t} = 0 \qquad \frac{\partial \bar{f}}{\partial \gamma} = 0 \qquad \bar{w} = 0 \quad (24)$$

Substituting Eq. 21 into Eq. 20, taking the mean time-average of Eq. 20, and applying the relationships given in Eq. 22, 23, and 24, the Reynolds equations for the axisymmetric, steady-state flow over a body of revolution are obtained.

(Flow along body)

$$\frac{\bar{u}}{(1+Ky)}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} + \frac{\bar{u}\bar{v}}{(1+Ky)}K + \frac{1}{(1+Ky)}\frac{\partial \overline{u'^2}}{\partial x} + \frac{\partial \overline{u'v'}}{\partial y}$$

$$+ \frac{\overline{u'^2}}{r(1+Ky)}\frac{\partial r}{\partial x} + \frac{2\overline{u'v'}}{(1+Ky)}K + \frac{\overline{u'v'}}{r}\cos\beta - \frac{\overline{w'^2}}{r(1+Ky)}\frac{\partial r}{\partial x}$$

$$= -\frac{1}{\rho(1+Ky)}\frac{\partial \bar{p}}{\partial x} - \nu\left[\frac{1}{(1+Ky)}\frac{\partial^2 \bar{v}}{\partial x \partial y} + \frac{\cos\beta}{r(1+Ky)}\frac{\partial \bar{v}}{\partial x} - \frac{K}{(1+Ky)^2}\frac{\partial \bar{v}}{\partial x}\right.$$

$$\left. - \frac{K}{(1+Ky)}\frac{\partial \bar{u}}{\partial y} - \frac{K\cos\beta}{r(1+Ky)}\bar{u} + \frac{K^2}{(1+Ky)^2}\bar{u} - \frac{\partial^2 \bar{u}}{\partial y^2} - \frac{\cos\beta}{r}\frac{\partial \bar{u}}{\partial y}\right]$$

(Flow normal to body)

$$\bar{v}\frac{\partial \bar{v}}{\partial y} + \frac{\bar{u}}{(1+Ky)}\frac{\partial \bar{v}}{\partial x} - \frac{\bar{u}^2}{(1+Ky)}K + \frac{\partial \overline{v'^2}}{\partial y} + \frac{1}{(1+Ky)}\frac{\partial \overline{u'v'}}{\partial x}$$

$$+ \frac{\overline{u'v'}}{r(1+Ky)}\frac{\partial r}{\partial x} - \frac{\overline{u'^2}}{(1+Ky)}K + \frac{\overline{v'^2}}{(1+Ky)}K + \frac{\overline{v'^2}}{r}\cos\beta - \frac{\overline{w'^2}}{r}\cos\beta$$

$$= -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial y} - \nu\left[-\frac{1}{(1+Ky)^2}\frac{\partial^2 \bar{v}}{\partial x^2} - \frac{1}{r(1+Ky)^2}\frac{\partial r}{\partial x}\frac{\partial \bar{v}}{\partial x} + \frac{y}{(1+Ky)^3}\frac{\partial K}{\partial x}\frac{\partial \bar{v}}{\partial x}\right.$$

$$+ \frac{K}{(1+Ky)^2}\frac{\partial \bar{u}}{\partial x} + \frac{\bar{u}}{(1+Ky)^2}\frac{\partial K}{\partial x} + \frac{\bar{u}K}{r(1+Ky)^2}\frac{\partial r}{\partial x} - \frac{\bar{u}Ky}{(1+Ky)^3}\frac{\partial K}{\partial x}$$

$$\left. + \frac{1}{(1+Ky)}\frac{\partial^2 \bar{u}}{\partial y \partial x} + \frac{1}{r(1+Ky)}\frac{\partial r}{\partial x}\frac{\partial \bar{u}}{\partial y}\right]$$

(Flow around body)

$$\frac{1}{(1+Ky)}\frac{\partial\overline{u'w'}}{\partial x}+\frac{\partial\overline{v'w'}}{\partial y}+\frac{2\overline{u'w'}}{r(1+Ky)}\frac{\partial r}{\partial x}+\frac{\overline{v'w'}}{(1+Ky)}K+\frac{2\overline{v'w'}}{r}\cos\beta=0$$

(Continuity)

$$\frac{\partial\bar{u}}{\partial x}+\frac{\bar{u}}{r}\frac{\partial r}{\partial x}+(1+Ky)\frac{\partial\bar{v}}{\partial y}+\bar{v}K+\frac{(1+Ky)\cos\beta}{r}\bar{v}=0 \qquad (25)$$

BOUNDARY-LAYER EQUATIONS

In order to apply Eq. 25 in a practical manner to the solution of physical problems, a great simplification is necessary. This can be accomplished, following after Prandtl (Ref. 12), by considering the flow as composed of two regions: a region close to the body where the viscous and turbulent stresses strongly affect the flow (the boundary layer) and a region outside the boundary layer where the flow is essentially potential. The fluid velocity in the x direction at the edge of the boundary layer is denoted as U, and for elongated bodies of practical interest (excluding the region around the nose where laminar flow normally exists) there results

$$\frac{U}{U_o}=O(1) \qquad (26)$$

where U_o is the undisturbed free-stream velocity. A length L may be defined by the relationship

$$\left(\frac{\partial\bar{u}}{\partial x}\right)_{max}=O\left(\frac{U_o}{L}\right) \qquad (27)$$

For most practical problems, the length, L, so defined will be of the order of magnitude of the length of the body and will satisfy the condition

$$\frac{\delta}{L}<<1 \qquad (28)$$

The radius of curvature, 1/K, of the body surface in the meridian plane will normally be of the order of magnitude of the body length over that portion (excluding the region around the nose where laminar flow normally exists) of practical bodies that are in turbulent flow. Hence it is not unreasonable to assume that

$$K = O\left(\frac{1}{L}\right) \tag{29}$$

Since this development must be applicable to the case where the boundary-layer thickness is not necessarily small compared to the body radius, it must be assumed that

$$r = O(\delta) \tag{30}$$

Applying Eq. 26 through 30, and using the normal order of magnitude considerations for determining the boundary layer equations from the Navier-Stokes equations, the relative magnitudes of the terms in Eq. 25 involving the mean velocity components can be determined. However, such an analysis will tell nothing about the order of magnitude of the terms involving the mean time-average of the products and squares of the fluctuation velocity components. To obtain some idea of the magnitude of these terms it is necessary to resort to experimental data.

Some of the most reliable measurements of the fluctuation velocity components in a turbulent boundary layer approaching separation have been made by Newman (Ref. 13). Using Eq. 27, an appropriate value of L was determined for the second series of tests in Ref. 13. The data from this second series was then analyzed and the following orders of magnitude determined.

$$\begin{aligned}
&\frac{\overline{u'v'}}{U_o^2} = O\left(\frac{\delta}{L}\right) \qquad \frac{d\left(\dfrac{\overline{u'v'}}{U_o^2}\right)}{d\left(\dfrac{x}{L}\right)} = O\left(\frac{\delta}{L}\right) \qquad \frac{d\left(\dfrac{\overline{u'v'}}{U_o^2}\right)}{d\left(\dfrac{y}{L}\right)} = O(1) \\
&\frac{\overline{u'^2}}{U_o^2} = O\left(\frac{\delta}{L}\right) \qquad \frac{d\left(\dfrac{\overline{u'^2}}{U_o^2}\right)}{d\left(\dfrac{x}{L}\right)} = O\left(\frac{\delta}{L}\right) \qquad \frac{d\left(\dfrac{\overline{u'^2}}{U_o^2}\right)}{d\left(\dfrac{y}{L}\right)} = O(1) \\
&\frac{\overline{v'^2}}{U_o^2} = O\left(\frac{\delta}{L}\right) \qquad \frac{d\left(\dfrac{\overline{v'^2}}{U_o^2}\right)}{d\left(\dfrac{x}{L}\right)} = O\left(\frac{\delta}{L}\right) \qquad \frac{d\left(\dfrac{\overline{v'^2}}{U_o^2}\right)}{d\left(\dfrac{y}{L}\right)} = O(1)
\end{aligned} \tag{31}$$

Since Newman's experiments did not include a measurement of $\overline{w'^2}$, it was necessary to examine some other data. The experiments of Ref. 14, 15, and 16 indicate that the values of $\overline{w'^2}$ tend to lie between those of $\overline{v'^2}$ and $\overline{u'^2}$. Hence it is reasonable to assume that

$$\frac{\overline{w'^2}}{U_o^2} = O\left(\frac{\delta}{L}\right) \qquad \frac{d\left(\dfrac{\overline{w'^2}}{U_o^2}\right)}{d\left(\dfrac{x}{L}\right)} = O\left(\frac{\delta}{L}\right) \qquad \frac{d\left(\dfrac{\overline{w'^2}}{U_o^2}\right)}{d\left(\dfrac{y}{L}\right)} = O(1) \tag{32}$$

With the addition of Eq. 31 and 32, a complete order of magnitude analysis of Eq. 25 is possible. In making such an analysis, it is customary to divide the first three parts of Eq. 25 by U_o^2/L and the last part by U_o/L, putting them in nondimensional form. Since inertia and viscous effects are of the same order of magnitude in the boundary layer, an examination of the inertia and viscous terms in this nondimensional form shows that

$$\nu = O\left(\frac{U_o \delta^2}{L}\right)$$

All the terms will then be of the order of magnitude 1, δ/L, $(\delta/L)^2$, $(\delta/L)^3$, $(\delta/L)^4$. Retaining only those terms of the order of magnitude 1, there results

(Flow along body)

$$\frac{\overline{u}}{U_o}\frac{\partial\left(\dfrac{\overline{u}}{U_o}\right)}{\partial\left(\dfrac{x}{L}\right)} + \frac{\overline{v}}{U_o}\frac{\partial\left(\dfrac{\overline{u}}{U_o}\right)}{\partial\left(\dfrac{y}{L}\right)} = -\frac{\partial\left(\dfrac{\overline{p}}{\rho U_o^2}\right)}{\partial\left(\dfrac{x}{L}\right)}$$

$$+\ \nu\frac{1}{r/L}\frac{\partial\left[\dfrac{r}{L}\dfrac{\partial(\overline{u}/U_o)}{\partial(y/L)}\right]}{\partial\left(\dfrac{y}{L}\right)} - \frac{1}{r/L}\frac{\partial\left(\dfrac{r}{L}\dfrac{\overline{u'v'}}{U_o^2}\right)}{\partial\left(\dfrac{y}{L}\right)} \tag{33}$$

(Flow normal to body)

$$\left(\frac{\bar{u}}{U_o}\right)^2 KL - \frac{1}{r/L}\,\frac{\partial\left(\frac{r}{L}\,\frac{\overline{v'^2}}{U_o^2}\right)}{\partial\left(\frac{y}{L}\right)} + \frac{\frac{\overline{w'^2}}{U_o^2}}{r/L}\cos\beta = \frac{\partial\left(\frac{\bar{p}}{\rho U_o^2}\right)}{\partial\left(\frac{y}{L}\right)} \tag{34}$$

(Continuity)

$$\frac{1}{r/L}\,\frac{\partial\left(\frac{r}{L}\,\frac{\bar{u}}{U_o}\right)}{\partial\left(\frac{x}{L}\right)} + \frac{1}{r/L}\,\frac{\partial\left(\frac{r}{L}\,\frac{\bar{v}}{U_o}\right)}{\partial\left(\frac{y}{L}\right)} = 0 \tag{35}$$

The equation for flow around the body has been dropped since it involves terms containing only the fluctuation velocities.

Equation 34 expresses the variation of static pressure across the boundary layer. Integrating Eq. 34 with respect to y/L and taking the partial derivative with respect to x/L, there results

$$\frac{\partial\left(\frac{\bar{p}}{\rho U_o^2}\right)}{\partial\left(\frac{x}{L}\right)} = \frac{\partial\left(\frac{\bar{p}}{\rho U_o^2}\right)_{y=\delta}}{\partial\left(\frac{x}{L}\right)} + \frac{\partial}{\partial\left(\frac{x}{L}\right)}\int_{\delta/L}^{y/L} KL\left(\frac{\bar{u}}{U_o}\right)^2 d\left(\frac{y}{L}\right)$$

$$- \frac{\partial\left(\frac{\overline{v'^2}}{U_o^2}\right)}{\partial\left(\frac{x}{L}\right)} - \int_{\delta/L}^{y/L} \frac{\partial\left\{\left[\left(\frac{\overline{v'^2}}{U_o^2}\right)\Big/\left(\frac{r}{L}\right)\right]\cos\beta\right\}}{\partial\left(\frac{x}{L}\right)} d\left(\frac{y}{L}\right)$$

$$+ \int_{\delta/L}^{y/L} \frac{\partial\left\{\left[\left(\frac{\overline{w'^2}}{U_o^2}\right)\Big/\left(\frac{r}{L}\right)\right]\cos\beta\right\}}{\partial\left(\frac{x}{L}\right)} d\left(\frac{y}{L}\right) \tag{36}$$

Equation 36 may now be substituted into Eq. 33; however, the magnitude of the terms should be examined first. The first term on the right side of Eq. 36 expresses the pressure gradient at the edge of the boundary layer arising from the external, essentially potential flow. The remaining terms express the variation of the pressure gradient across the boundary layer. An order of magnitude analysis of these remaining terms shows[7] that they are of the order δ/L. Hence, the retaining of these remaining terms in Eq. 36 when substituting into Eq. 33 would be inconsistent with the approximation already made when terms of the order of δ/L were discarded in arriving at Eq. 33. Therefore, within the approximation of retaining terms of the order of magnitude 1 in the nondimensional Reynolds equations, it may be assumed that the pressure is constant across the boundary layer at the value given by the external flow at the edge of the boundary layer ($y = \delta$). Since the external flow is assumed to be essentially potential, Bernoulli's equation applies, so that

$$\frac{\partial\left(\dfrac{\overline{p}}{\rho U_o^2}\right)}{\partial\left(\dfrac{x}{L}\right)} = \frac{\partial\left(\dfrac{\overline{p}}{\rho U_o^2}\right)_{y=\delta}}{\partial\left(\dfrac{x}{L}\right)} = -\frac{U}{U_o}\,\frac{d\left(\dfrac{U}{U_o}\right)}{d\left(\dfrac{x}{L}\right)}$$

Substituting the expression above into Eq. 33 and returning to dimensional form, the first-order boundary-layer equations for the turbulent, axisymmetric flow over a body of revolution are obtained.

$$\overline{u}\,\frac{\partial\overline{u}}{\partial x} + \overline{v}\,\frac{\partial\overline{u}}{\partial y} = U\,\frac{dU}{dx} + \nu\,\frac{1}{r}\,\frac{\partial\left(r\,\dfrac{\partial\overline{u}}{\partial y}\right)}{\partial y} - \frac{1}{r}\,\frac{\partial(r\overline{u'v'})}{\partial y}$$

$$\frac{1}{r}\,\frac{\partial(r\overline{u})}{\partial x} + \frac{1}{r}\,\frac{\partial(r\overline{v})}{\partial y} = 0 \qquad (37)$$

It has been pointed out by several investigators (e. g., Ref. 13 and 17) that the above first-order boundary-layer equations in their integrated form (the momentum integral equation) do not yield satisfactory results in a strong adverse pressure gradient when the shear stress at

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[7] This result is expected since the terms in Eq. 34 are of the order of magnitude 1 so that an integration across the boundary layer with respect to y/L yield a variation in pressure across the boundary of the order δ/L.

the wall is being determined from a knowledge of the pressure gradient and the velocity profiles. It is pointed out that certain terms of the order of δ/L (namely, the turbulent normal stress terms in the equation for flow along the body and the terms expressing the variation in pressure across the boundary layer in the equation for flow normal to the body) must be retained in the nondimensional Reynolds equation to yield anything approaching a satisfactory solution. This is true, because in a strong adverse pressure gradient, the shear stress term is very small compared to the terms expressing the pressure gradient and the rate of change of the velocity profiles. Hence, the magnitude of a small term in an equation containing large terms is being sought where a number of small terms have already been discarded from the equation. In effect, the determination of a term of second-order magnitude is being attempted using a first-order equation.

In the present work, it is planned to use the first-order boundary-layer equations to determine the velocity profiles from a knowledge of the pressure gradient and the shear at the wall. Thus a first-order effect is being solved for using a first-order equation, and there is good reason to believe that satisfactory results essentially to the point of separation can be obtained.

The expressions for the laminar shear stress, τ_ℓ, and the turbulent shear stress, τ_t, are

$$\tau_\ell = \mu \frac{\partial \bar{u}}{\partial y}$$
$$\tau_t = -\rho\overline{u'v'} \tag{38}$$

Letting τ equal the total shear stress ($\tau = \tau_\ell + \tau_t$) and substituting Eq.38 into Eq. 37, there is obtained

$$\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} = U\frac{dU}{dx} + \frac{1}{\rho}\frac{1}{r}\frac{\partial (r\tau)}{\partial y} \tag{39}$$

$$\frac{1}{r}\frac{\partial (r\bar{u})}{\partial x} + \frac{1}{r}\frac{\partial (r\bar{v})}{\partial y} = 0 \tag{40}$$

These are the first-order boundary-layer equations in their most compact form for the axisymmetric flow over a body of revolution.

MOMENTUM INTEGRAL EQUATION

The integration of Eq. 39 across (x = constant) the boundary layer (from y = 0 to y = δ) can be carried out by making use of the last of Eq. 17 and by using the continuity equation 40 to eliminate $\overline{v}$. There results

$$\frac{d\theta_r}{dx} + (2 + H_r)\frac{\theta_r}{U}\frac{dU}{dx} + \frac{\theta_r}{r_o}\frac{dr_o}{dx} = \frac{\tau_o}{\rho U^2} = \left(\frac{v_*}{U}\right)^2 \tag{41}$$

where

$$H_r = \frac{\delta_r^*}{\theta_r} \tag{42}$$

$$\delta_r^* = \int_o^\delta \left(1 - \frac{\overline{u}}{U}\right)dy + \frac{\cos\beta}{r_o}\int_o^\delta \left(1 - \frac{\overline{u}}{U}\right)ydy \tag{43}$$

$$\theta_r = \int_o^\delta \frac{\overline{u}}{U}\left(1 - \frac{\overline{u}}{U}\right)dy + \frac{\cos\beta}{r_o}\int_o^\delta \frac{\overline{u}}{U}\left(1 - \frac{\overline{u}}{U}\right)ydy \tag{44}$$

Equation 41 is the momentum integral equation for the axisymmetric flow over a body of revolution. δ_r^* and θ_r must be looked upon as pseudo-displacement and pseudo-momentum thicknesses since they do not retain the physical significance their two-dimensional counterparts have. If $1/r_o$ is set equal to zero in Eq. 41, 43, and 44, Eq. 41 reduces to the more familiar two-dimensional momentum integral equation of von Karman (Ref. 18) and δ_r^* and θ_r reduce to the displacement thickness and momentum thickness of two-dimensional flow.

ENERGY INTEGRAL EQUATION

Another integral relationship can be determined from Eq. 39 by first multiplying Eq. 39 by $\overline{u}$ and then carrying out the integration across the boundary layer in a manner similar to that used in obtaining Eq. 41. The resulting expression is

$$\frac{d\delta_r^{**}}{dx} + 3\frac{\delta_r^{**}}{U}\frac{dU}{dx} + \frac{\delta_r^{**}}{r_o}\frac{dr_o}{dx}$$

$$= \frac{2}{\rho U^3}\int_o^\delta \tau\frac{\partial\overline{u}}{\partial y}dy + \frac{2}{\rho U^3}\frac{\cos\beta}{r_o}\int_o^\delta \tau\frac{\partial\overline{u}}{\partial y}ydy \tag{45}$$

where

$$\delta_r^{**} = \int_0^\delta \frac{\bar{u}}{U}\left(1 - \frac{\bar{u}^2}{U^2}\right)dy + \frac{\cos\beta}{r_o}\int_0^\delta \frac{\bar{u}}{U}\left(1 - \frac{\bar{u}^2}{U^2}\right)ydy \qquad (46)$$

Equation 45 is the energy integral equation for the axisymmetric turbulent flow over a body of revolution, and δ_r^{**} is a pseudo-energy thickness since it does not retain the physical significance of its two-dimensional counterpart. If $1/r_o$ is set equal to zero in Eq. 45 and 46, Eq. 45 reduces to the two-dimensional energy integral equation first proposed by Wieghardt (Ref. 19) in a slightly different form for laminar flow, and δ_r^{**} reduces to the energy thickness of two-dimensional flow. A convenient thickness ratio to use in conjunction with the energy integral equation is

$$\overline{H}_r = \frac{\delta_r^{**}}{\theta_r} \qquad (47)$$

EVALUATION OF THE THICKNESS PARAMETERS

Using the identities

$$\frac{\bar{u}}{U}\left(1 - \frac{\bar{u}}{U}\right) = \left(1 - \frac{\bar{u}}{U}\right) - \left(1 - \frac{\bar{u}}{U}\right)^2$$

$$\frac{\bar{u}}{U}\left(1 - \frac{\bar{u}^2}{U^2}\right) = 2\left(1 - \frac{\bar{u}}{U}\right) - 3\left(1 - \frac{\bar{u}}{U}\right)^2 + \left(1 - \frac{\bar{u}}{U}\right)^3$$

the expressions for the thickness parameters, Eq. 43, 44, and 46 may be written in the following form.

$$\frac{\delta_r^*}{\delta} = \frac{v_*}{U}\int_0^1 \frac{U-\bar{u}}{v_*}\,d\left(\frac{y}{\delta}\right) + \frac{v_*}{U}\frac{\delta}{r_o}\cos\beta\int_0^1 \frac{U-\bar{u}}{v_*}\frac{y}{\delta}\,d\left(\frac{y}{\delta}\right)$$

$$\frac{\theta_r}{\delta} = \frac{v_*}{U}\int_0^1 \frac{U-\bar{u}}{v_*}\,d\left(\frac{y}{\delta}\right) + \frac{v_*}{U}\frac{\delta}{r_o}\cos\beta\int_0^1 \frac{U-\bar{u}}{v_*}\frac{y}{\delta}\,d\left(\frac{y}{\delta}\right)$$

$$- \left(\frac{v_*}{U}\right)^2\int_0^1 \left(\frac{U-\bar{u}}{v_*}\right)^2 d\left(\frac{y}{\delta}\right) - \left(\frac{v_*}{U}\right)^2\frac{\delta}{r_o}\cos\beta\int_0^1 \left(\frac{U-\bar{u}}{v_*}\right)^2\frac{y}{\delta}\,d\left(\frac{y}{\delta}\right)$$

$$\frac{\delta_r^{**}}{\delta} = 2\frac{v_*}{U}\int_0^1 \frac{U-\bar{u}}{v_*}\,d\left(\frac{y}{\delta}\right) + 2\,\frac{v_*}{U}\,\frac{\delta}{r_o}\cos\beta\int_0^1 \frac{U-\bar{u}}{v_*}\frac{y}{\delta}\,d\left(\frac{y}{\delta}\right)$$

$$- 3\left(\frac{v_*}{U}\right)^2\int_0^1 \left(\frac{U-\bar{u}}{v_*}\right)^2 d\left(\frac{y}{\delta}\right) - 3\left(\frac{v_*}{U}\right)^2\frac{\delta}{r_o}\cos\beta\int_0^1 \left(\frac{U-\bar{u}}{v_*}\right)^2\frac{y}{\delta}\,d\left(\frac{y}{\delta}\right)$$

$$+ \left(\frac{v_*}{U}\right)^3\int_0^1 \left(\frac{U-\bar{u}}{v_*}\right)^3 d\left(\frac{y}{\delta}\right) + \left(\frac{v_*}{U}\right)^3\frac{\delta}{r_o}\cos\beta\int_0^1 \left(\frac{U-\bar{u}}{v_*}\right)^3\frac{y}{\delta}\,d\left(\frac{y}{\delta}\right)$$

Using the analytical approximation, Eq. 14, to Coles' two-parameter mean velocity profiles, the integrals in the expressions above may be evaluated (see Appendix). The expressions for H_r, $\overline{H}_r$, and δ then become

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$$H_r = \frac{\frac{\delta_r^*}{\delta}}{\frac{\theta_r}{\delta}}$$

$$H_r = \frac{N_1 + AN_2 + \frac{\delta}{r_o} \cos}{N_1 + AN_2 + \frac{\delta}{r_o}\cos\beta(N_3 + AN_4) - \frac{v_*}{U}\frac{1}{\kappa}\Big[N_5 + AN}$$

$$\overline{H}_r = \frac{\frac{\delta_r^{**}}{\delta}}{\frac{\theta_r}{\delta}}$$

$$\overline{H}_r = \frac{2\left[N_1 + AN_2 + \frac{\delta}{r_o}\cos\beta(N_3 + AN_4)\right] - \frac{v_*}{U}\frac{3}{\kappa}\left[N_5 + AN_6 + A^2N_7 + \frac{\delta}{r_o}\cos\beta(N_8 + AN_9 + A^2N_{10})\right] + \Big(}{N_1 + AN_2 + \frac{\delta}{r_o}\cos\beta(N_3 + N_4) - \frac{v_*}{U}\frac{1}{\kappa}\Big[N_5 + AN_6}$$

$$\delta = \frac{\theta_r}{\frac{v_*}{U}\frac{1}{\kappa}\left[N_1 + AN_2 + \frac{\delta}{r_o}\cos\beta(N_3 + N_4)\right] - \left(\frac{v_*}{U}\right)^2 \frac{1}{\kappa^2}\Big[N}$$

where the numerical constants, N_1 through N_{18}, are given in the Appe

$$\cdots s\,\beta(N_3 + AN_4)$$

$$\overline{\cdots N_6 + A^2N_7 + \frac{\delta}{r_o}\cos\beta(N_8 + AN_9 + A^2N_{10})\Big]} \qquad (48)$$

$$\left(\frac{v_*}{U}\right)^2 \frac{1}{\kappa^2}\left[N_{11} + AN_{12} + A^2N_{13} + A^3N_{14} + \frac{\delta}{r_o}\cos\beta(N_{15} + AN_{16} + A^2N_{17} + A^3N_{18})\right]$$

$$\overline{\cdots 6 + A^2N_7 + \frac{\delta}{r_o}\cos\beta(N_8 + AN_9 + A^2N_{10})\Big]} \qquad (49)$$

$$\overline{\cdots N_5 + AN_6 + A^2N_7 + \frac{\delta}{r_o}(N_8 + AN_9 + A^2N_{10})\Big]} \qquad (50)$$

…endix.

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THE SHEAR STRESS AT THE WALL

The shear stress at the wall, $\tau_o = \rho v_*^2$, is specified if the velocity at the edge of the boundary, U, the boundary-layer thickness, δ, and the profile parameter, A, are known. This is seen by applying the boundary condition $\overline{u} = U$ when $y = \delta$ to Eq. 13.

$$\frac{U}{v_*} = \frac{1}{\kappa} \ln \frac{\delta v_*}{\nu} + C \frac{1}{\kappa}\left[-\frac{1}{m} + A\left(1 - \frac{n}{m}\right)\right]$$

This equation can be put in a more useful form in which the boundary layer thickness, δ, plays a secondary role by using Eq. 50. The following implicit equation for v_*/U results.

$$\begin{aligned} \frac{U}{v_*} &= \frac{1}{\kappa} \ln \frac{\theta_r U}{\nu} + C + \frac{1}{\kappa}\left[-\frac{1}{m} + A\left(1 - \frac{n}{m}\right)\right] \\ &\quad - \frac{1}{\kappa} \ln\left\{\frac{1}{\kappa}\left[N_1 + AN_2 + \frac{\delta}{r_o} \cos\beta(N_3 + AN_4)\right]\right. \\ &\quad \left. - \frac{v_*}{U}\frac{1}{\kappa^2}\left[N_5 + AN_6 + A^2N_7 + \frac{\delta}{r_o} \cos\beta(N_8 + AN_9 + A^2N_{10})\right]\right\} \end{aligned} \tag{51}$$

DETERMINATION OF THE ENERGY DISSIPATION

The two integrals

$$\frac{2}{\rho U^3}\int_o^\delta \tau \frac{\partial \overline{u}}{\partial y} dy + \frac{2}{\rho U^3}\frac{\cos\beta}{r_o}\int_o^\delta \tau \frac{\partial \overline{u}}{\partial y} y dy = \frac{2}{\rho U^3}\frac{1}{r_o}\int_o^\delta r\tau \frac{\partial \overline{u}}{\partial y} dy \tag{52}$$

on the right side of the energy integral equation (45), represent the rate at which organized kinetic energy in the boundary layer is dissipated into random kinetic energy (turbulence) and heat. In order to evaluate these integrals, a knowledge of the variation of the shear stress, τ, across the boundary layer is needed. This variation can be found by using a momentum integral equation in which the integration is not carried completely across the boundary layer ($y = 0$ to $y = \delta$) but rather partially across the boundary layer ($y = 0$ to $y = y$). Using the first-order boundary-layer equations, 39 and 40, the incompletely integrated momentum integral equation takes the form

$$\frac{1}{\rho} r\tau = \frac{1}{\rho} r_o\tau_o - \bar{u}\int_o^y \frac{\partial(r\bar{u})}{\partial x} dy$$
$$+ \int_o^y \bar{u}\frac{\partial(r\bar{u})}{\partial x} dy + \int_o^y r\bar{u}\frac{\partial \bar{u}}{\partial x} dy - U\frac{dU}{dx}\int_o^y r dy \qquad (53)$$

This result reduces to the momentum integral equation 41, if evaluated at $y = \delta$.

The question arises at this point whether the first-order equation 53 can yield a reasonably good solution for the variation of τ across the boundary layer or whether second-order effects must be included. Coles (Ref. 1) compared the variation in τ across the boundary layer given by the two-dimensional form[8] of Eq. 53 with Newman's second series of tests (Ref. 13). The integrals in the two-dimensional form of Eq. 53 were evaluated using the mean velocity profiles given by Eq. 5 which had been fit to the experimental mean velocity profiles. Coles found that the general behavior of τ versus y was reasonably well predicted but that both the boundary condition $\tau = \tau_o$ when $y = 0$ and $\tau = 0$ when $y = \delta$ could not always be satisfied since the experimental data did not conform to the momentum integral equation at all points in the flow. This noncomformity was attributed primarily to a deviation of the flow from true two-dimensionality. By a small alteration in the continuity equation, Coles made the calculated values of τ satisfy both boundary conditions and brought them in good agreement with the experimental values. Since it is planned to solve the momentum integral equation simultaneously with the energy integral equation, Eq. 53 will satisfy both the boundary conditions $\tau = \tau_o$ when $y = 0$ and $\tau = 0$ when $y = \delta$, and based on Coles' findings should give a satisfactory solution for τ versus y.

Substituting Eq. 53 into Eq. 52, the energy dissipation integrals may be written, dropping the multiplier $2/U^3$,

$$\frac{1}{\rho}\int_o^\delta \tau\frac{\partial \bar{u}}{\partial y} dy + \frac{1}{\rho}\frac{\cos\beta}{r_o}\int_o^\delta \tau\frac{\partial \bar{u}}{\partial y} y dy$$
$$= \frac{1}{\rho} r_o\tau_o\int_o^\delta \frac{\partial \bar{u}}{\partial y} dy - \int_o^\delta \left(\bar{u}\frac{\partial \bar{u}}{\partial y}\int_o^y \frac{\partial(r\bar{u})}{\partial x} dy\right) dy$$
$$+ \int_o^\delta \left(\frac{\partial \bar{u}}{\partial y}\int_o^y \bar{u}\frac{\partial(r\bar{u})}{\partial x} dy\right) dy + \int_o^\delta \left(\frac{\partial \bar{u}}{\partial y}\int_o^y r\bar{u}\frac{\partial \bar{u}}{\partial x} dy\right) dy$$
$$- U\frac{dU}{dx}\int_o^\delta \left(\frac{\partial \bar{u}}{\partial y}\int_o^y r dy\right) dy \qquad (54)$$

[8] The two-dimensional form of Eq. 53 is obtained by dividing through by r and letting $r_o \to \infty$.

Using the last of Eq. 17 and integration by parts, Eq. 54 can be expanded into a form containing much simpler integrals. In this operation, one approximation is indicated based on the previous order of magnitude analysis. Differentiating the last of Eq. 17 with respect to x, there results

$$\frac{\partial r}{\partial x} = \frac{dr_o}{dx} - y \sin\beta \frac{d\beta}{dx}$$

But from Fig. 3 it is seen that

$$\sin\beta = \frac{dr_o}{dx} \qquad \frac{d\beta}{dx} = -K$$

Therefore,

$$\frac{\partial r}{\partial x} = \frac{dr_o}{dx}(1 + yK)$$

But $y_{max} = \delta$ and $K = O(1/L)$ so that $yK = O(\delta/L)$, and since $\delta/L << 1$ it follows that

$$\frac{\partial r}{\partial x} \cong \frac{dr_o}{dx}$$

Equation 54 then takes on the following form in which the multiplier $2/U^3$ has been reintroduced and y has been nondimensionalized by δ.

$$\frac{2}{\rho U^3}\int_o^\delta \tau \frac{\partial \overline{u}}{\partial y} dy + \frac{2}{\rho U^3}\frac{\cos\beta}{r_o}\int_o^\delta \tau \frac{\partial \overline{u}}{\partial y} y dy$$

$$= \frac{2\tau_o}{\rho U^2} - \frac{\delta}{r_o}\frac{dr_o}{dx}\int_o^1 \frac{\overline{u}}{U} d\left(\frac{y}{\delta}\right) - \frac{\delta}{U}\int_o^1 \frac{\partial \overline{u}}{\partial x} d\left(\frac{y}{\delta}\right)$$

$$- \frac{\delta}{r_o}\cos\beta \frac{\delta}{U}\int_o^1 \frac{\partial \overline{u}}{\partial x}\frac{y}{\delta} d\left(\frac{y}{\delta}\right) - \frac{\delta}{r_o}\frac{dr_o}{dx}\int_o^1 \left(\frac{\overline{u}}{U}\right)^3 d\left(\frac{y}{\delta}\right)$$

$$- 3\frac{\delta}{U}\int_o^1 \left(\frac{\overline{u}}{U}\right)^2 \frac{\partial \overline{u}}{\partial x} d\left(\frac{y}{\delta}\right) - 3\frac{\delta}{r_o}\cos\beta \frac{\delta}{U}\int_o^1 \left(\frac{\overline{u}}{U}\right)^2 \frac{\partial \overline{u}}{\partial x}\frac{y}{\delta} d\left(\frac{y}{\delta}\right)$$

$$+ 2\frac{\delta}{r_o}\frac{dr_o}{dx}\int_o^1 \left(\frac{\overline{u}}{U}\right)^2 d\left(\frac{y}{\delta}\right) + 4\frac{\delta}{U}\int_o^1 \frac{\overline{u}}{U}\frac{\partial \overline{u}}{\partial x} d\left(\frac{y}{\delta}\right)$$

$$+ 4\frac{\delta}{r_o}\cos\beta\frac{\delta}{U}\int_o^1 \frac{\overline{u}}{U}\frac{\partial\overline{u}}{\partial x}\frac{y}{\delta}d\left(\frac{y}{\delta}\right) - 2\frac{\delta}{U}\frac{dU}{dx}\int_o^1 d\left(\frac{y}{\delta}\right)$$

$$- 2\frac{\delta}{U}\frac{dU}{dx}\frac{\delta}{r_o}\cos\beta\int_o^1 \frac{y}{\delta}d\left(\frac{y}{\delta}\right) + 2\frac{\delta}{U}\frac{dU}{dx}\int_o^1 \frac{\overline{u}}{U}d\left(\frac{y}{\delta}\right)$$

$$+ 2\frac{\delta}{U}\frac{dU}{dx}\frac{\delta}{r_o}\cos\beta\int_o^1 \frac{\overline{u}}{U}\frac{y}{\delta}d\left(\frac{y}{\delta}\right) \qquad (55)$$

The $\partial\overline{u}/\partial x$ is readily determined from Eq. 13.

$$\frac{\partial\overline{u}}{\partial x} = \frac{\overline{u}}{U}\frac{U}{v_*}\frac{dv_*}{dx} + \frac{1}{\kappa}\frac{dv_*}{dx} + \frac{v_*}{\kappa}\left(\frac{dA}{dx} - \frac{An}{\delta}\frac{d\delta}{dx}\right)\left(\frac{y}{\delta}\right)^n$$

$$+ \frac{v_*}{\kappa}\left[-\frac{n}{m}\frac{dA}{dx} + \frac{(1 + An)}{\delta}\frac{d\delta}{dx}\right]\left(\frac{y}{\delta}\right)^m \qquad (56)$$

Using Eq. 56 and the following identities,

$$\frac{\overline{u}}{U} = 1 - \frac{v_*}{U}\left(\frac{U - \overline{u}}{v_*}\right)$$

$$\left(\frac{\overline{u}}{U}\right)^2 = 1 - 2\frac{v_*}{U}\left(\frac{U - \overline{u}}{v_*}\right) + \left(\frac{v_*}{U}\right)^2\left(\frac{U - \overline{u}}{v_*}\right)^2$$

$$\left(\frac{\overline{u}}{U}\right)^3 = 1 - 3\frac{v_*}{U}\left(\frac{U - \overline{u}}{v_*}\right) + 3\left(\frac{v_*}{U}\right)^2\left(\frac{U - \overline{u}}{v_*}\right)^2 - \left(\frac{v_*}{U}\right)^3\left(\frac{U - \overline{u}}{v_*}\right)^3$$

all the integrals on the right side of Eq. 55 may be put into the form

$$\int_o^1 \left(\frac{U - \overline{u}}{v_*}\right)^q\left(\frac{y}{\delta}\right)^s d\left(\frac{y}{\delta}\right) \qquad (q = \text{integer})$$

These integrals may be evaluated using Eq. 14 (see Appendix) and yield the final expression for the energy dissipation integrals.

$$
\begin{aligned}
&\frac{2}{\rho U^3}\int_o^\delta \tau \frac{\partial \bar{u}}{\partial y}\,dy + \frac{2}{\rho U^3}\,\frac{\cos\beta}{r_o}\int_o^\delta \tau \frac{\partial \bar{u}}{\partial y}\,y\,dy \\
&\quad = 2\Big(\frac{v_*}{U}\Big)^2 + 2\,\frac{\delta}{U}\,\frac{dv_*}{dx}\,\frac{1}{\kappa}\Bigg[N_1 + AN_2 + \frac{\delta}{r_o}\cos\beta(N_3 + AN_4) \\
&\qquad + \frac{v_*}{U}\,\frac{1}{\kappa}\Bigg\{N_1 - \frac{5}{2}N_5 + A\Big(N_2 - \frac{5}{2}N_6\Big) - A^2\,\frac{5}{2}N_7 \\
&\qquad + \frac{\delta}{r_o}\cos\beta\Big[N_3 - \frac{5}{2}N_8 + A\Big(N_4 - \frac{5}{2}N_9\Big) - A^2\,\frac{5}{2}N_{10}\Big]\Bigg\} \\
&\qquad + \frac{3}{2}\Big(\frac{v_*}{U}\Big)^2\,\frac{1}{\kappa^2}\Bigg\{N_{11} - N_5 + A(N_{12} - N_6) + A^2(N_{13} - N_7) + A^3N_{14} \\
&\qquad + \frac{\delta}{r_o}\cos\beta[N_{15} - N_8 + A(N_{16} - N_9) + A^2(N_{17} - N_{10}) + A^3N_{18}]\Bigg\}\Bigg] \\
&\qquad + 2\,\frac{dA}{dx}\,\frac{v_*}{U}\,\frac{\delta}{\kappa}\Bigg[\frac{v_*}{U}\,\frac{1}{\kappa}\Bigg\{N_{19} - \frac{n}{m}N_{23} + A\Big(N_{20} - \frac{n}{m}N_{24}\Big) \\
&\qquad + \frac{\delta}{r_o}\cos\beta\Big[N_{21} - \frac{n}{m}N_{25} + A\Big(N_{22} - \frac{n}{m}N_{26}\Big)\Big]\Bigg\} \\
&\qquad + \frac{3}{2}\Big(\frac{v_*}{U}\Big)^2\,\frac{1}{\kappa^2}\Bigg\{\frac{n}{m}N_{33} - N_{27} + A\Big(\frac{n}{m}N_{34} - N_{28}\Big) + A^2\Big(\frac{n}{m}N_{35} - N_{29}\Big) \\
&\qquad + \frac{\delta}{r_o}\cos\beta\Big[\frac{n}{m}N_{36} - N_{30} + A\Big(\frac{n}{m}N_{37} - N_{31}\Big) + A^2\Big(\frac{n}{m}N_{38} - N_{32}\Big)\Big]\Bigg\}\Bigg] \\
&\qquad + 2\,\frac{d\delta}{dx}\,\frac{v_*}{U}\,\frac{1}{\kappa}\Bigg[\frac{v_*}{U}\,\frac{1}{\kappa}\Bigg\{N_{23} + A(N_{24} + nN_{23} - nN_{19}) + A^2(nN_{24} - nN_{20}) \\
&\qquad + \frac{\delta}{r_o}\cos\beta[N_{25} + A(N_{26} + nN_{25} - nN_{21}) + A^2(nN_{26} - nN_{22})]\Bigg\}
\end{aligned}
$$

$$+ \frac{3}{2}\left(\frac{v_*}{U}\right)^2 \frac{1}{\kappa^2}\left\{-N_{33} + A(nN_{27} - nN_{33} - N_{34})\right.$$

$$+ A^2(nN_{28} - nN_{34} - N_{35}) + A^3(nN_{29} - nN_{35})$$

$$+ \frac{\delta}{r_o} \cos\beta[-N_{36} + A(nN_{30} - nN_{36} - N_{37})$$

$$\left.\left. + A^2(nN_{31} - nN_{37} - N_{38}) + A^3(nN_{32} - nN_{38})]\right\}\right]$$

$$+ \frac{\delta}{r_o}\frac{dr_o}{dx}\frac{v_*}{U}\frac{1}{\kappa}\left[\frac{v_*}{U}\frac{1}{\kappa}(-N_5 - AN_6 - A^2N_7)\right.$$

$$\left. + \left(\frac{v_*}{U}\right)^2 \frac{1}{\kappa^2}(N_{11} + AN_{12} + A^2N_{13} + A^3N_{14})\right]$$

$$+ 2\frac{\delta}{U}\frac{dU}{dx}\frac{v_*}{U}\frac{1}{\kappa}\left[-N_1 - AN_2 + \frac{\delta}{r_o}\cos\beta(-N_3 - AN_4)\right] \qquad (57)$$

where the numerical constants N_1 through N_{38} are given in the Appendix and n and m are regarded as equal to 2.50 and 2.75, as mentioned previously. Also the substitution $\tau_o/\rho U^2 = (v_*/U)^2$ has been made.

THE BOUNDARY-LAYER SOLUTION

Equations 41, 42, 45, 47, 48, 49, 50, 51, and 57 constitute a set of nine differential, algebraic, and transcendental equations in the nine unknowns:

θ_r

δ_r^*

H_r

δ_r^{**}

$\overline{H}_r$

δ

v_*

A

$$\frac{2}{\rho U^3}\int_o^\delta \tau \frac{\partial \overline{u}}{\partial y} dy + \frac{2}{\rho U^3}\frac{\cos\beta}{r_o}\int_o^\delta \tau \frac{\partial \overline{u}}{\partial y} y dy$$

It is assumed the pressure distribution at the outer edge of the boundary layer $U = f(x)$ and the body shape $1/r_o = f(x)$ are known. A solution of these equations for $1/r_o \neq 0$ is a first-order solution for the turbulent boundary layer on a body of revolution in axisymmetric flow. For $1/r_o = 0$, it is a first-order solution for the turbulent flow on a two-dimensional body. As brought out earlier in the development, both solutions are restricted to the case where the radius of curvature, $1/K$, of the body surface in the direction of flow is large compared to the boundary-layer thickness.

An iterative method for obtaining a simultaneous solution of these nine equations is under development, and programming of the iterative method on the IBM 7090 will be undertaken in the near future.

To commence such an iterative solution, starting values of θ_r, H_r, and x must be known. Starting values of θ_r and x may be determined from a laminar boundary-layer solution from the stagnation point to the transition point. The Pohlhausen method (Ref. 20) offers a convenient way of obtaining the laminar solution since it may be applied in a simple approximate form to both the two-dimensional body (Ref. 21) and the body of revolution (Ref. 22). A reliable starting value of H_r is not easily obtained because of the scarcity of information on the change of this thickness ratio through transition, particularly in a pressure gradient. It is necessary, therefore, to determine a starting value of H_r by some rather uncertain means such as is suggested in Ref. 23 and 24. This is admittedly a weak link in the boundary-layer solution.

THE DISPLACEMENT THICKNESS ON A BODY OF REVOLUTION

For the axisymmetric flow on a body of revolution, δ_r^* is a pseudo-displacement thickness, as mentioned earlier. It does not represent the increase in the body dimension (normal to the surface) that would displace the streamlines of a totally potential flow around the body by the same amount they are displaced as a result of the retardation of the flow in the boundary layer. The true displacement thickness, δ^*, which does represent this condition, is determined, as pointed out by Granville (Ref. 25) by the relation

$$\delta^* = \frac{-1 + \sqrt{1 + 2\dfrac{1}{r_o}\cos\beta\,\delta_r^*}}{\dfrac{1}{r_o}\cos\beta}$$

In the limit as $1/r_o \to 0$, the equation above yields $\delta^* = \delta_r^*$, which is the expected result from the definition of δ_r^*.

It is often customary to compute the potential pressure distribution around a body increased in dimension by the displacement thickness. This is done to account for, in part, the effect of the boundary layer on the potential flow outside the boundary layer. To carry out this calculation, the values of δ^* are needed.

DISCUSSION AND CONCLUSIONS

Although most of the development leading from the Navier-Stokes equations to the momentum and energy integral equations can be found in various references, it was included in this report in order to emphasize the assumptions and restrictions underlying the first-order boundary-layer equations and to give a line of unbroken continuity to the development.

The foregoing development has shown that the use of Coles' universal, two-parameter, mean velocity profiles made possible a scheme for obtaining an essentially exact solution of the first-order turbulent boundary-layer equations in integrated form. This was possible because these profiles inherently contain sufficient information to determine not only the shear stress at the wall but also the variation of shear stress across the boundary layer, making possible the determination of the rate of energy dissipation in the boundary layer. Due to the fairly rigorous development of this method and the fact that velocity profiles can be accurately represented essentially to the point of separation, it is anticipated that an improved technique for estimating the inflow velocity profiles to propellers and the skin-friction drag of bodies of revolution will result.

Appendix
EVALUATION OF INTEGRALS

The development in the body of this report requires the evaluation of integrals of the form

$$\int_0^1 \left(\frac{U - \bar{u}}{v_*}\right)^q \left(\frac{y}{\delta}\right)^s d\left(\frac{y}{\delta}\right) \qquad (q = \text{integer})$$

using Eq. 14 (repeated below)

$$\frac{U - \bar{u}}{v_*} = -\frac{1}{\kappa} \ln \frac{y}{\delta} + \frac{A}{\kappa}\left[1 - \left(\frac{y}{\delta}\right)^n\right] - \frac{1}{\kappa}\left(\frac{1}{m} + \frac{n}{m}A\right)\left[1 - \left(\frac{y}{\delta}\right)^m\right] \qquad (14)$$

where $n = 2.50$ and $m = 2.75$. These integrations are straightforward but they are rather long and tedious when q is greater than one. In carrying out these integrations, the terms involving $\ln(y/\delta)$ lead to improper integrals since Eq. 14 does not properly describe the flow in the region very near the wall. These improper integrals are, however, convergent, and the effect on the value of these integrals due to the deviation from the true flow in the transition zone and laminar sublayer is not significant.

$$\kappa \int_0^1 \frac{U - \bar{u}}{v_*} d\left(\frac{y}{\delta}\right) = N_1 + AN_2$$

$$\kappa \int_0^1 \frac{U - \bar{u}}{v_*} \frac{y}{\delta} d\left(\frac{y}{\delta}\right) = N_3 + AN_4$$

$$\kappa^2 \int_0^1 \left(\frac{U - \bar{u}}{v_*}\right)^2 d\left(\frac{y}{\delta}\right) = N_5 + AN_6 + A^2N_7$$

$$\kappa^2 \int_0^1 \left(\frac{U - \bar{u}}{v_*}\right)^2 \frac{y}{\delta} d\left(\frac{y}{\delta}\right) = N_8 + AN_9 + A^2N_{10}$$

$$\kappa^3 \int_0^1 \left(\frac{U - \bar{u}}{v_*}\right)^3 d\left(\frac{y}{\delta}\right) = N_{11} + AN_{12} + A^2N_{13} + A^3N_{14}$$

$$\kappa^3 \int_0^1 \left(\frac{U - \bar{u}}{v_*}\right)^3 \frac{y}{\delta} d\left(\frac{y}{\delta}\right) = N_{15} + AN_{16} + A^2N_{17} + A^3N_{18}$$

$$\kappa \int_0^1 \frac{U - \bar{u}}{v_*} \left(\frac{y}{\delta}\right)^{2.50} d\left(\frac{y}{\delta}\right) = N_{19} + AN_{20}$$

$$\kappa \int_0^1 \frac{U - \bar{u}}{v_*} \left(\frac{y}{\delta}\right)^{3.50} d\left(\frac{y}{\delta}\right) = N_{21} + AN_{22}$$

$$\kappa \int_0^1 \frac{U - \bar{u}}{v_*} \left(\frac{y}{\delta}\right)^{2.75} d\left(\frac{y}{\delta}\right) = N_{23} + AN_{24}$$

$$\kappa \int_0^1 \frac{U - \bar{u}}{v_*} \left(\frac{y}{\delta}\right)^{3.75} d\left(\frac{y}{\delta}\right) = N_{25} + AN_{26}$$

$$\kappa^2 \int_0^1 \left(\frac{U - \bar{u}}{v_*}\right)^2 \left(\frac{y}{\delta}\right)^{2.50} d\left(\frac{y}{\delta}\right) = N_{27} + AN_{28} + A^2N_{29}$$

$$\kappa^2 \int_0^1 \left(\frac{U - \bar{u}}{v_*}\right)^2 \left(\frac{y}{\delta}\right)^{3.50} d\left(\frac{y}{\delta}\right) = N_{30} + AN_{31} + A^2N_{32}$$

$$\kappa^2 \int_0^1 \left(\frac{U - \bar{u}}{v_*}\right)^2 \left(\frac{y}{\delta}\right)^{2.75} d\left(\frac{y}{\delta}\right) = N_{33} + AN_{34} + A^2N_{35}$$

$$\kappa^2 \int_0^1 \left(\frac{U - \bar{u}}{v_*}\right)^2 \left(\frac{y}{\delta}\right)^{3.75} d\left(\frac{y}{\delta}\right) = N_{36} + AN_{37} + A^2N_{38}$$

where

$N_1 = 0.733333334$	$N_{20} = 0.004761904$
$N_2 = 0.047619048$	$N_{21} = 0.018731375$
$N_3 = 0.144736842$	$N_{22} = 0.002736728$
$N_4 = 0.014619884$	$N_{23} = 0.030085470$
$N_5 = 1.406495726$	$N_{24} = 0.004102564$
$N_6 = 0.116197456$	$N_{25} = 0.016251154$
$N_7 = 0.003296703$	$N_{26} = 0.002419842$
$N_8 = 0.128485687$	$N_{27} = 0.016054940$
$N_9 = 0.018400596$	$N_{28} = 0.003292518$
$N_{10} = 0.000792209$	$N_{29} = 0.000186741$
$N_{11} = 4.182609023$	$N_{30} = 0.005999780$
$N_{12} = 0.36895053$	$N_{31} = 0.001405893$
$N_{13} = 0.01447839$	$N_{32} = 0.000088639$
$N_{14} = 0.00025215$	$N_{33} = 0.012292719$
$N_{15} = 0.185477054$	$N_{34} = 0.002619501$
$N_{16} = 0.030226715$	$N_{35} = 0.000153119$
$N_{17} = 0.001986392$	$N_{36} = 0.004834318$
$N_{18} = 0.000051483$	$N_{37} = 0.001163109$
$N_{19} = 0.035918367$	$N_{38} = 0.000074911$

These values are doubtful in the last one or two significant digits.

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ABSTRACT. A method for calculating the incompressible turbulent boundary layer based on the "law of the wall" and Coles' "law of the wake" is presented. The method is applicable to two-dimensional bodies and to bodies of revolution in axisymmetric flow when the boundary-layer thickness is not necessarily small compared to the body radius. The carrying out of this method involves a simultaneous solution of the momentum integral equation and the energy integral equation, assuming that the mean velocity profiles are given by a universal two-parameter representation as suggested

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NAVWEPS Report 8510

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IN REPLY REFER TO:
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4 June 1965

From: Commander, U. S. Naval Ordnance Test Station
To: Distribution List of NAVWEPS Report 8510

Subj: NAVWEPS Report 8510, A Turbulent Boundary-Layer Calculation Method Based on the Law of the Wall and the Law of the Wake, dated November 1964; transmittal of addendum for

Encl: (1) Addendum dated 5 May 1965

1. It is requested that the discussion provided on the attached enclosure be incorporated in NAVWEPS Report 8510.

C. E. Van Hagan
C. E. VAN HAGAN
By direction

ADDENDUM

It should be pointed out that an additional condition must be imposed on the turbulent boundary-layer solution. Experimental evidence shows that for flow on a flat plate with zero pressure gradient, the profile parameter Π in Eq. 5 or the profile parameter A in Eq. 13 remains essentially a constant. This fact was first noted by Schultz-Grunow (Ref. 3). Therefore, the solution must yield $\frac{dA}{dx} = 0$ for A = Flat plate value, $\frac{dU}{dx} = 0$, and $\frac{dr_o}{dx} = 0$. (The flow on a cylinder of constant radius is similar to that on a flat plate if $\frac{\delta}{r_o} \ll 1$.) This requirement of the solution can be obtained by adding a correction term to the energy dissipation function.

A discussion of this point had been postponed for a future report since it was not certain in what form the correction term would be incorporated into the solution, and the computer program that would be required to investigate the various possibilities was not available when NAVWEPS Report 8510 was published. However, it was completely overlooked that the application of this requirement to the solution was not only desirable but was essential to make the equations independent. The fact that the equations mentioned on page 36 contain a redundancy was pointed out in a communication from Dr. G. E. Gadd of the National Physical Laboratory, Teddington, England. Since the variation of τ across the boundary layer used in the energy integral equation was determined from the partially integrated momentum equation, the momentum and energy integral equations are not independent. A more complete discussion of this area will appear in a forthcoming report that will present the method used to solve the equations and comparisons of computer solutions with experimental data.

5 May 1965

Enclosure (1)